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Spring 2022, Howard Math 274:
Combinatorics II (2nd semester intro grad comb.)

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Class info:
- Meets MWF 11:10-12, online via Zoom
- Office hrs: by appointment (email me!)
- Text: "Combinatorics: the Art of Counting" by Sagan (pdf linked to on website)
  Last semester: Ch.s 1-5, this semester: Ch.s 6-8
- Grading: There are 3 HW's (Feb., March, April)
  Beyond that, I expect you to show up to and participate in class (ask questions!)
- Disclaimer: Last semester's class was a version of a class I had taught before. This semester is new for me (may be rough around edges)

What is this class about?

We will continue our investigation/enumeration of discrete structures with a new focus on

symmetries = algebra!

The major topics (in order) will be:

(1/4 semester) group actions on combinatorial objects

(3/4 semester) symmetric functions

Let's start with group actions now!

Q: How many ways are there to color the vertices of a square w/ 3 colors (red, blue, green)?

Of course, we know **A**: $3^4 = 81$

But... square has some <u>symmetries</u> and we might want to take these into account. We might want to consider <u>rotations</u> of same coloring the same:

$$\begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array} \;=\; \begin{array}{ccc} B & - & R \\ | & & | \\ G & - & R \end{array} \;=\; \begin{array}{ccc} G & - & B \\ | & & | \\ R & - & R \end{array} \;=\; \begin{array}{ccc} R & - & G \\ | & & | \\ R & - & B \end{array}$$

In this case, there will certainly be fewer than 81 colorings. We might also want to consider <u>reflections</u> of colorings to be the same:

$$\begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array} \;=\; \begin{array}{ccc} B & - & G \\ | & & | \\ R & - & R \end{array}$$

Depending on which symmetries we allow, we will get a different # of colorings.

To systematize these kinds of "counting up to symmetry" problems, we will review the algebraic notions of <u>groups</u> and <u>group actions</u>.

DEF'N A <u>group</u> G is a set G w/ a binary operation, <u>multiplication</u> $\cdot : G \times G \to G$

s.t.
(1) (associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in G$,
(2) (identity) there exists an <u>identity element</u> $e \in G$ for which $g \cdot e = e \cdot g = g$ $\forall g \in G$
(3) (inverses) $\forall g \in G$, there exists an inverse $g^{-1} \in G$ with $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Hopefully you've seen this def'n before, but where does it come from? Important notion: group action!

DEF'N Let G be a group and X a set. An action of G on X (sometimes denoted $G \curvearrowright X$) is an assignment of a map $g: X \to X$ for each $g \in G$ s.t. $(gh)(x) = g(h(x))$ $\forall\, g,h \in G,\, x \in X$ and $e(x) = x$ $\forall\, x \in X$, where $e \in G$ is identity.
From now on we restrict to finite G and X!

E.g. The symmetric group S_n of permutations of $[n] := \{1, 2, \ldots, n\}$:

$$S_n = \left\{ \sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \right\}$$ "two-line notation"

has a canonical action on $[n]$.

$n=4$ $\quad \begin{pmatrix} 1&2&3&4 \\ 2&1&4&3 \end{pmatrix} \cdot 2 = 1$

$$\Bigl(\underset{g}{\begin{pmatrix} 1&2&3&4 \\ 2&1&4&3 \end{pmatrix}} \underset{h}{\begin{pmatrix} 1&2&3&4 \\ 1&3&4&2 \end{pmatrix}} \Bigr) \cdot 2 = \underset{gh}{\begin{pmatrix} 1&2&3&4 \\ 2&4&3&1 \end{pmatrix}} \cdot 2 = 4 \checkmark$$

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E.g. Symmetric group S_n also acts on subsets of $[n]$ in a natural way: for $A \subseteq [n]$, $\sigma \in S_n$

$$\sigma \cdot A := \{ \sigma(a) : a \in A \}$$

$$\begin{pmatrix} 1&2&3&4 \\ 2&1&4&3 \end{pmatrix} \cdot \{2,3\} = \{1,4\}$$

E.g. If $H \leq G$ is a subgroup (i.e., a subset closed under the multiplication) and G acts on X, then there is an induced action of H on X.

symmetric gp. on finite set X

Rmk: Every action $G \curvearrowright X$ is (essentially) induced from $G \leq S_X$, (more precisely, from some quotient H of G $\leq S_X$)

since ...

Prop: If $G \curvearrowright X$ is an action then:

(1) $g: X \to X$ is a bijection (i.e., permutation)

(2) $e: X \to X$ is the identity map.

Pf: Exercise. See Sagan. $\square$

One more kind of action will be very important for us:

E.g. If $G \curvearrowright X$ then $G \curvearrowright Y^X$ for any set Y

where $Y^X := \{ \text{all functions } f: X \to Y \}$

by the rule $g \cdot f(x) = f(g^{-1} \cdot x)$ $\forall x \in X, f \in Y^X$.

(The g^{-1} is to make the action satisfy $(gh)f = g(hf)$, but can be confusing at first...)

E.g. Let $X = \{1,2,3,4\}$ which we think of as vertices of square, and $Y = \{R, G, B\}$ colors. Then $f: X \to Y$, is a coloring:

$$f: \begin{array}{ccc} 1 & - & 2 \\ | & & | \\ 4 & - & 3 \end{array} \longrightarrow \begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array}$$

Let $G = \langle (1,2,3,4) \rangle = \{ e, (1,2,3,4), (1,3)(2,4), (1,4,3,2) \}$

(subgroup generated by cycle $(1,2,3,4)$)

action Y^X:

$$(1,2,3,4) \cdot \begin{array}{ccc} R & - & R \\ | & & | \\ B & - & G \end{array} = \begin{array}{ccc} B & - & R \\ | & & | \\ G & - & R \end{array} \circlearrowright$$

= rotations of colorings of square!

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Now that we see how "counting up to symmetry" connects to group actions, we need one more definition:

<u>DEF'N</u> Let $G \curvearrowright X$ and $x \in X$. The <u>orbit</u> of x, denoted $\mathcal{O}_x$, is $\mathcal{O}_x := \{g \cdot x : g \in G\}$.

"Everything I can get to from x."

Prop. The orbits $\mathcal{O}_x, x \in X$ <u>partition</u> the set X.

<u>E.g.</u> Let $G = \langle \sigma = (1,2,3,4) \rangle \subseteq S_4$ act on $X = \{\text{size-2 subsets of } \{1,2,3,4\}\}$

$$\{1,2\} \xrightarrow{\sigma} \{2,3\} \xrightarrow{\sigma} \{3,4\} \xrightarrow{\sigma} \{1,4\} \xrightarrow{\sigma} \{1,2\} \qquad \{1,3\} \underset{\sigma}{\overset{\sigma}{\rightleftarrows}} \{2,4\}$$

there are two orbits of G acting on X

(<u>Note</u>: A group generated by a single element is called <u>cyclic</u>. A finite cyclic group is $\simeq (\mathbb{Z}/N\mathbb{Z}, +)$ for some N.)

To answer "counting up to symmetry" problems, we want to know <u>how</u> <u>many</u> <u>orbits</u> does an action $G \curvearrowright X$ have? <u>Burnside's Lemma</u> will give answer.

First we need one basic result in group theory.

<u>DEF'N</u> Let $G \curvearrowright X$ and $x \in X$. The <u>stabilizer</u> of x, denoted G_x is

$$G_x := \{g \in G : g \cdot x = x\}.$$

E.g. With G, X as in previous example, and $x = \{1,3\}$, $G_x = \{e, (1,3)(2,4)\} \leq G$.

Prop. Any stabilizer G_x is a subgroup of G.

Pf: If $g \cdot x = x$ and $h \cdot x = x$ then $(gh) \cdot x = g(h(x)) = g(x) = x$. ☒

Thm (Orbit-Stabilizer Theorem)

1/26 For any $x \in X$, $\# \mathcal{O}_x \cdot \# G_x = \# G$.

Pf: Recall that for a subgroup $H \leq G$, a (left) coset of H is a set $gH := \{gh : h \in H\}$ for some $g \in G$.

Notation $G/H = \{gH : g \in G\}$ = set of cosets of H in G.

Claim $\exists$ bijection $\varphi : G/G_x \to \mathcal{O}_x$ given by $\varphi(g G_x) = g \cdot x$.

Pf: Need to check well-definedness: for $h \in G_x$,

$\varphi((gh) G_x) = (gh) \cdot x = g(h(x)) = g \cdot x$. ✓

Bijectiveness: φ^{-1} given by $\varphi^{-1}(g \cdot x) = g G_x$. Well-defined since if $g \cdot x = h \cdot x$, $h^{-1} g \in G_x$ so $hG_x = gG_x$ ☒

To finish proof, use another basic gp. theory result:

Thm (Lagrange's Thm) $\# G/H = \# G / \# H$.

Pf: $\exists$ bijection $\phi : H \to gH$ for any $gH \in G/H$
$\phi : h \mapsto gh$

So all gH have same size $\Rightarrow$ must be $\#G/\#H$ of them. ☒

So $\# \mathcal{O}_x = \# G/G_x = \# G / \# G_x$. ✓

E.g.: Let $G = \langle (1,2,3,4) \rangle$ and $X = \{\text{size 2 subsets of } [4]\}$ as before. Then

with $x = \{1,3\}$, $\#\mathcal{O}_x \cdot \#G_x = 2 \cdot 2 = 4 = \#G$ ✓

w/ $x' = \{1,2\}$, $\#\mathcal{O}_x \cdot \#G_x = 4 \cdot 1 = 4 = \#G$ ✓

Now we are ready to give formula for # of orbits:

<u>Lemma</u> ("Burnside's Lemma")

The number of orbits of an action $G \curvearrowright X$ is

$$\frac{1}{\#G} \sum_{g \in G} \#X^g,$$

where $X^g := \{x \in X : g(x) = x\}$ is the <u>fixed-point</u> <u>set</u> of $g \in G$.

<u>Pf</u>: Note that for any integer k,

$$\underbrace{\frac{1}{k} + \frac{1}{k} + \cdots + \frac{1}{k}}_{k \text{ times}} = 1$$

Hence for any orbit $\mathcal{O}$ of $G \curvearrowright X$ we have

$$\sum_{x \in \mathcal{O}} \frac{1}{\#\mathcal{O}_x} = \sum_{x \in \mathcal{O}} \frac{1}{\#\mathcal{O}} = 1$$

So that

$$\underset{\text{of } G \curvearrowright X}{\#\text{ of orbits}} = \sum_{x \in X} \frac{1}{\#\mathcal{O}_x}.$$

By the orbit-stabilizer Thm $\frac{1}{\#\mathcal{O}_x} = \frac{\#G_x}{\#G}$, so

$$\#\text{ of orbits} = \frac{1}{\#G} \sum_{x \in X} \#G_x.$$

Now we want to change from summing over $x \in X$ to summing over $g \in G$...

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To do that, consider the $\#G \times \#X$ matrix M whose (g,x) entry is $M_{(g,x)} = \begin{cases} 1 & \text{if } g(x)=x \\ 0 & \text{otherwise} \end{cases}$

e.g. with $G = \langle \sigma = (1,2,3,4) \rangle$, $X = \{\text{2-subsets of } [4]\}$,

$$M = \begin{array}{c|cccccc} & \{1,2\} & \{2,3\} & \{3,4\} & \{1,4\} & \{1,3\} & \{2,4\} \\ \hline e & 1 & 1 & 1 & 1 & 1 & 1 \\ \sigma & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma^2 & 0 & 0 & 0 & 0 & 1 & 1 \\ \sigma^3 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Note that $\#G_x$ is sum of column of M corresponding to x.

So $\sum_{x \in X} \#G_x$ = sum of all columns = sum of all entries of M.

But $\#X^g$ is sum of row of M corresponding to g.

So $\sum_{g \in G} \#X^g$ = sum of all rows = sum of all entries of M = $\sum_{x \in X} \#G_x$.

Thus # of orbits $= \frac{1}{\#G} \sum_{g \in G} \#X^g$, as claimed. $\square$

E.g. with $G = \langle \sigma = (1,2,3,4) \rangle$, $X = \{\text{2-subsets of } [4]\}$ as before

$$\frac{1}{\#G} \sum_{g \in G} \#X^g = \frac{1}{4}\left(\#X^e + \#X^\sigma + \#X^{\sigma^2} + \#X^{\sigma^3}\right)$$

$$= \frac{1}{4}(6+0+2+0) = \frac{1}{4} \cdot 8$$

$$= 2 = \#\text{ orbits of } G \curvearrowright X. \checkmark$$

But what we really wanted to count was # orbits of colorings $G \curvearrowright Y^X$ induced from action $G \curvearrowright X$.

Prop. Let $G \curvearrowright X$ and consider induced action $G \curvearrowright Y^X$. Then for any $g \in G$, $f \in (Y^X)^g \Leftrightarrow f(x) = f(g^{-1} \cdot x)\ \forall x \in X$.

Pf: Recall $g \cdot f(x) = f(g^{-1}x)$, hence $gf = f$
$\Leftrightarrow f(x) = gf(x)\ \forall x \in X \Leftrightarrow f(x) = f(g^{-1}x)\ \forall x \in X$. ☒

When $G \curvearrowright X$, each $g \in G$ determines a permutation $g : X \to X$ and hence has an associated cycle structure.

Let $\boxed{c(g) := \#\text{ cycles of perm. } g : X \to X.}$

Prop. For any $g \in G$, $\#(Y^X)^g = (\#Y)^{c(g)}$.

Pf: To determine a coloring $f \in (Y^X)^g$, i.e., f w/ $f(x) = f(g^{-1} \cdot x)$ $\forall x \in X$, must choose for each cycle of g one color $y \in Y$ to give to all elts of that cycle.

Eg. w/ $g = (1,3,4)(2)(5,6)$ Choose
- color for 1, 3, 4
- color for 2
- color for 5, 6.

Total # of choices $= \underbrace{\#Y \cdot \#Y \cdots \#Y}_{\text{each cycle}} = \#Y^{c(g)}$. ☒

Cor ("Unweighted Pólya counting")

Let $G \curvearrowright X$ and consider induced coloring action $G \curvearrowright Y^X$.

Then # orbits of $G \curvearrowright Y^X$ $= \dfrac{1}{\#G} \sum_{g \in G} (\#Y)^{c(g)}$.

E.g. We can now answer our motivating Q: How many colorings of vertices of square, with 3 colors, up to rotation?

Take $G = \langle \sigma = (1,2,3,4) \rangle \curvearrowright X = [4]$ w/ colors set $Y = \{R, G, B\}$. Then # orbits $G \curvearrowright Y^X$

$$= \frac{1}{\#G} \sum_{g \in G} (\#Y)^{c(g)} = \frac{1}{4}\left((\#Y)^{c(e)} + (\#Y)^{c(\sigma)} + (\#Y)^{c(\sigma^2)} + (\#Y)^{c(\sigma^3)} \right)$$

$$= \frac{1}{4}\left(3^4 + 3^1 + 3^2 + 3^1\right) = \frac{1}{4}(96) = \underline{24}.$$

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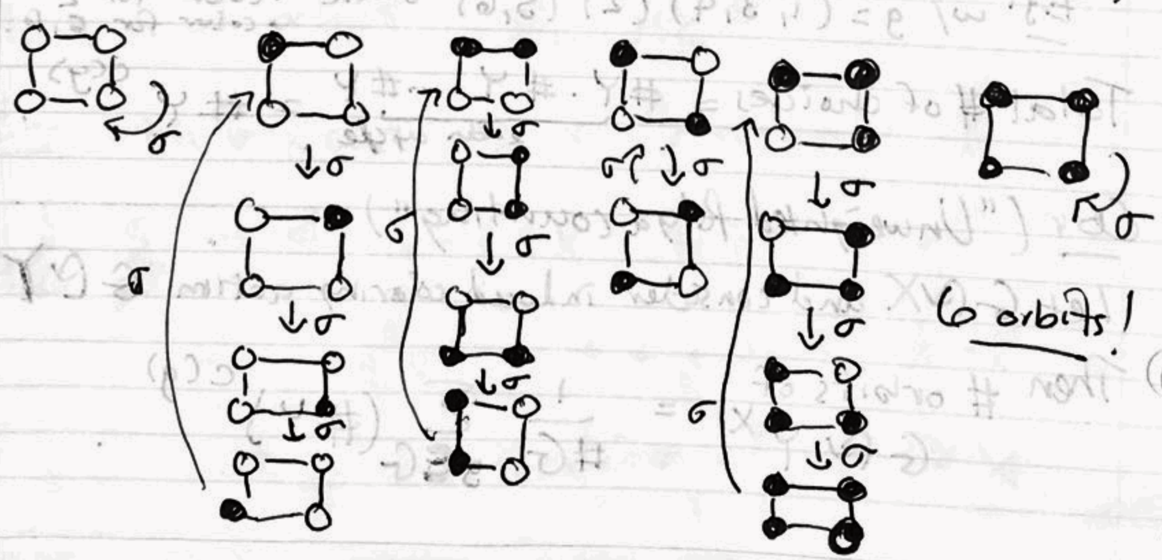
Let's take same G, X as last example, w/ $k = 2$.

Then #colorings up to symmetry $= \frac{1}{4}\left(2^4 + 2^1 + 2^2 + 2^1\right)$
$= \frac{1}{4}(24) = 6.$

This is small enough that we can check:

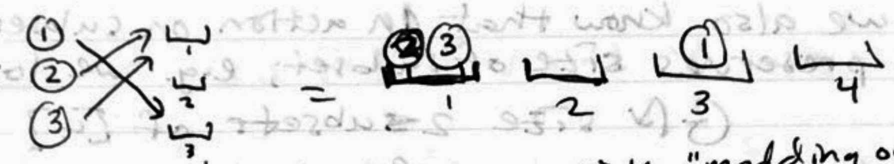


E.g. for a different kind of example, let's take $G = S_n$ full symmetric gp. acting on $X = [n]$ (in natural way),

and $Y = [k]$. In this case,

$$Y^X = \{\text{functions } f : [n] \to [k]\}$$
$$= \{\text{ways of putting } n \text{ labeled balls into } k \text{ labeled bins}\}.$$

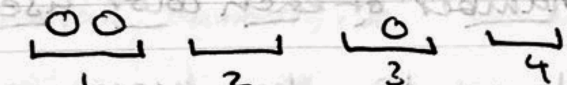
e.g.: f:



and $\{\text{orbits of } G \curvearrowright Y^X\}$ ← think: "modding out by S_n action on ball labels"

$$= \{\text{ways of putting } n \text{ \underline{un}labeled balls into } k \text{ labeled bins}\}$$

e.g. $G \cdot f =$



Last semester we saw using "stars and bars" that $\#$ orbits of $G \curvearrowright Y^X = \binom{n+k-1}{n}$.

We can also see this formula from the unweighted Pólya counting, which says

$$\#\text{orbits } G \curvearrowright Y^X = \frac{1}{\#G} \sum_{g \in G} (\#Y)^{c(g)} = \frac{1}{n!} \sum_{\sigma \in S_n} k^{c(\sigma)}$$

$$= \frac{1}{n!} \sum_{j=1}^{n} c(n,j) \cdot k^j, \text{ where}$$

$c(n,j) = \#\{\text{perms } \sigma \text{ in } S_n \text{ w/ } j \text{ total cycles}\}$
$= \text{(unsigned) Stirling \#'s of 1st kind.}$

$\boxed{\text{Last semester}} \Rightarrow \sum_{j=1}^{n} c(n,j) t^j = t(t+1)\cdots(t+(n-1)).$

$$\Rightarrow \#\text{ orbits} = \frac{1}{n!} \cdot k(k+1)\cdots(k+(n-1)) = \binom{n+k-1}{n} //$$