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Standard Young Tableaux

The product formula for the g.f. of r.p.p.'s of shape λ is nice, but what about counting a finite combinatorial set?

DEF'N A Standard Young Tableau of shape λ is a filling of boxes of λ w/ the numbers $1, 2, \ldots, n := |\lambda|$, each appearing exactly once, so that numbers are strictly increasing along rows + down columns.

e.g. The 2 SYT's of shape $\lambda = (2,2)$ are $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$

Let $f^\lambda := \#\mathrm{SYT}$'s of shape λ. Note that also

$$f^\lambda = [x_1 x_2 \cdots x_n]\, s_\lambda(x_1, x_2, \ldots) \leftarrow \text{"coefficient of } x_1 \cdots x_n \text{ in Schur fn."}$$

since an SYT is the same as a semi standard tableau of content $= (1,1,1,\ldots,1)$.

Thm ("Hook Length Formula", Frame-Robinson-Thrall, 1954)

$$f^\lambda = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)} \quad \text{for any partition } \lambda \vdash n$$

e.g. w/ $\lambda = (2,2)$, hook lengths are $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline\end{array}$

so that $f^\lambda = 4!/3\cdot 2\cdot 2\cdot 1 = 2$ ✓

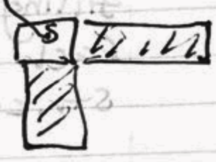
Note that $n! = \#$ ways to fill boxes of λ w/ numbers $1, 2, \ldots, n$ (each used once) w/out any requirement on order of #'s

So... HLF has a probabilistic interpretation:

It says that the probability a random filling is an SYT is exactly $\prod_{u \in \lambda} \frac{1}{h(u)}$.

Bogus probabilistic proof of HLF:

have to put smallest # here



- A filling is an SYT iff each entry is smallest among #'s in its hook.
- In a random filling, the probability that box u has entry smallest in its hook is $1/h(u)$
- So prob. random filling is SYT $= \prod_{u \in \lambda} 1/h(u)$

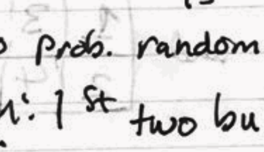
PROBLEM: 1st two bullets are correct, but can only take products for probabilities of independent events, and these events are very much not independent!

There is a valid probabilistic proof of HLF based on construction of a random SYT via "hook walk"

- Choose a random box u in λ to start at,
- unless we're at a SE border box, move to another random box in the hook of u
- when we hit a SE border box, put the number n there.

Then we repeat w/ where to put $n-1$, $n-2$, ... etc. down to 1.

e.g. In



we might start at u_1, then go to u_2, then go to u_3, and put $n = 15$ there

Main thing to show is that this procedure really produces each SYT w/ equal probability $\left(= \frac{\prod h(u)}{n!}\right)$

See Sagan §7.3 for proof of this... We'll give different proof! ...

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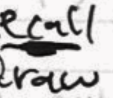
Instead, we will deduce HLF for SYTs from g.f. of r.p.p.'s! Actually, it will be easiest to explain this deduction in the more general setting of (finite) **posets**! (reflexive, antisymmetric, transitive)

Recall a poset is a set with a partial order. We draw posets using Hasse diagrams:

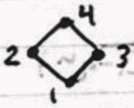
$P =$ 

$a < b,\ a < c,\ b < d,\ c < d$ (implied: $a < d$)

DEF'N A linear extension of a poset P is a list $p_1, p_2, \ldots, p_n$ of all elements s.t. $p_i \leq p_j \Rightarrow i \leq j$.
We let $\mathcal{L}(P) := \{\text{lin. ext.'s of } P\}$.

e.g. w/ P as above, $\mathcal{L}(P) = \{abcd, acbd\}$

DEF'N We say P is naturally labeled if elt's are $P = \{1, 2, \ldots, n\}$ and have $i \leq_P j \Rightarrow i \leq j$ (as numbers). In this case, we treat $\mathcal{L}(P) \subseteq S_n$ as a set of permutations.

e.g. $P =$ 

is nat. labeled and $\mathcal{L}(P) = \{1234, 1324\}$

RMK: Note that the identity $12\cdots n$ is always in $\mathcal{L}(P)$.

Recall that a descent of a permutation $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$ is a position $1 \leq i \leq n-1$ s.t. $\sigma_i > \sigma_{i+1}$. Set $D(\sigma) = \{\text{descents of } \sigma\}$ and recall that the major index of σ is

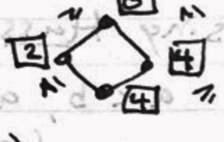
$$\operatorname{maj}(\sigma) := \sum_{i \in D(\sigma)} i.$$

e.g. $D(1234) = \emptyset$ and $D(1324) = \{2\}$ so that

$$\sum_{\sigma \in \mathcal{L}(P)} q^{\operatorname{maj}(\sigma)} = 1 + q^2,$$ where P is nat. labeled poset as above.

DEF'N A P-partition (for poset P) is a function $\pi : P \to \mathbb{N}$ that is order-reversing: i.e., $p \leq q \Rightarrow \pi(p) \geq \pi(q)$.

We use $|\pi| := \sum_{p \in P} \pi(p)$ (like w/ the r.p.p.'s).

e.g. One P-partition is 

$\boxed{\ } =$ value of π (as opposed to label)

Thm (G.f. for P-partitions)

For P naturally labeled, with $\#P = n$,
$$\sum_{\pi \text{ P-partition}} q^{|\pi|} = \frac{\sum_{\sigma \in \mathcal{L}(P)} q^{\mathrm{maj}(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

e.g. w/ P as before, g.f. is
$$= \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)(1-q^4)} = 1 + q + 3q^2 + 4q^3 + 7q^4 \cdots$$

Rmk: w/ $P = \underset{1}{\cdot} \; \underset{2}{\cdot} \cdots \underset{n}{\cdot}$ an n-element antichain

thm says
$$\underbrace{\sum_{\text{all } f : [n] \to \mathbb{N}} q^{|f|}}_{= \frac{1}{(1-q)^n}} = \frac{\sum_{\text{all } \sigma \in S_n} q^{\mathrm{maj}(\sigma)}}{(1-q)(1-q^2)\cdots(1-q^n)}$$
$$\Longleftrightarrow \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = [n]_q!,$$
something we proved last semester (maj and inv ~~equidistributed~~ same distr.).

In fact, proof we give will be same as last semester.

Cor For any poset P,
$$\#\mathcal{L}(P) = \lim_{q \to 1} \left(\sum_{\pi \text{ P-partition}} q^{|\pi|} \right) \cdot (1-q)(1-q^2)\cdots(1-q^n)$$

Pf: mult. both sides by $(1-q)(1-q^2)\cdots(1-q^n)$ in thm above, and take limit $q \to 1$ (or just plug in $q=1$). $\boxtimes$

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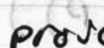
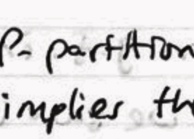
Before we prove the g.f. for P-partitions thm, let's explain how this corollary implies the HLF for SYTs. Basically we just need to match up the various terms.
To any partition $\lambda \vdash n$, associate poset P_λ ($\# P_\lambda = n$) where elt.'s are boxes, and $u \geq v \Leftrightarrow u$ northwest of v.

e.g. $\lambda =$  $\Leftrightarrow P_\lambda =$ 

With this construction, r.p.p.'s of sh$=\lambda$ $=$ P_λ-partitions and $\exists$ bij. between SYTs of sh$=\lambda$ and $\mathcal{L}(P_\lambda)$:

$$T \mapsto \text{box w/ } n, \text{ box w/ } n-1, \ldots, \text{ box w/ } 1$$

e.g. $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \mapsto d, e, b, c, a \in \mathcal{L}(P_\lambda)$

So by cor have for any $\lambda \vdash n$ that

$$f^\lambda = \# \text{SYTs of sh}=\lambda = \# \mathcal{L}(P_\lambda) = \lim_{q \to 1} (1-q)(1-q^2)\cdots(1-q^n) \cdot \sum_{\pi \text{ r.p.p. of sh}=\lambda} q^{|\pi|}$$

$$= \lim_{q \to 1} \frac{(1-q)(1-q^2)\cdots(1-q^n)}{\prod_{u \in \lambda}(1-q^{h(u)})} = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)}$$

g.f. of r.p.p.'s a.k.a. Hillman-Grassl

L'Hôpital's rule

Proving HLF!

Rmk By choosing a particular natural labeling of P_λ, can also obtain a q-analog of the HLF for SYTs this way, giving maj-g.f. of tableaux...

So now to finish everything, need to prove P-partition g.f.:

Pf (of P-partition g.f. in terms of $\mathcal{L}(P)$):
The idea (which we saw last semester!) is to break {P-partitions} into pieces corresponding to lin. ext.'s

Lemma Every $f: [n] \to \mathbb{N}$ has a unique $\sigma \in S_n$ such that f is σ-compatible in sense that

(write $f(j)=f_j$ for convenience) →
- $f_{\sigma_1} \geq f_{\sigma_2} \geq \cdots \geq f_{\sigma_n}$
- if $i \in D(\sigma)$ then $f_{\sigma_i} > f_{\sigma_{i+1}}$

This f is a P-partition $\Leftrightarrow \sigma \in \mathcal{L}(P)$.

Pf: Write $f_{\sigma_1} = f_{\sigma_2} = \cdots = f_{\sigma_a} > f_{\sigma_{a+1}} = \cdots = f_{\sigma_b} > f_{\sigma_{b+1}} = \cdots$
So that $\sigma_1 < \sigma_2 < \cdots < \sigma_a$ and $\sigma_{a+1} < \cdots < \sigma_b$ and ...

e.g. $\overset{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8}{f=(2,0,5,0,3,3,2,0)}$ has $\overset{5\ \ 3\ \ 3\ \ 2\ 2\ \ 0\ \ 0\ \ 0}{f_3 > f_5 = f_6 > f_1 = f_7 > f_2 = f_4 = f_8}$
so $\sigma = 3,5,6,1,7,2,4,8$ is unique perm. f is compatible with.
The statement about P-partition $\Leftrightarrow$ lin. ext. is clear. ☒

Thus,
$$\sum_{\pi \text{ P-partition}} q^{|\pi|} = \sum_{\sigma \in \mathcal{L}(P)} \sum_{\substack{f\ \sigma\text{-compatible}\\ \text{fun. } [n]\to\mathbb{N}}} q^{|f|}$$
$$= \sum_{\sigma \in \mathcal{L}(P)} \sum_{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)} q^{\mathrm{maj}(\sigma) + |\lambda|}$$

subtract off the <u>smallest</u> σ-compatible $f_0: [n] \to \mathbb{N}$

$$\begin{array}{lcl} & \begin{smallmatrix}3 & 5 & 6 & \bullet & 1 & 7 & \bullet & 2 & 4 & 8\end{smallmatrix} & \\ & (5,3,3,2,2,0,0,0) & = f \\ - & (2,2,2,1,1,0,0,0) & = f_0 \\ \hline & (3,1,1,1,1,0,0,0) & = \lambda \end{array}$$

$$= \sum_{\sigma \in \mathcal{L}(P)} q^{\mathrm{maj}(\sigma)} \cdot \sum_{\lambda: \ell(\lambda) \leq n} q^{|\lambda|}$$

NOTE: $\boxed{|f_0| = \mathrm{maj}(\sigma)}$
since $i \in D(\sigma)$
$\Rightarrow$ have to increase by one values in 1^{st} i spots to get a ~~descent~~ strict decrease

$$= \sum_{\sigma \in \mathcal{L}(P)} q^{\mathrm{maj}(\sigma)} \cdot \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$ ✓

☒