Graph coloring,
Math 4707, Spring 2021

Recall that for a graph G, the *chromatic number* of G, denoted $\chi(G)$, is the smallest number of colors needed to (properly) color the vertices of G. Graphs with $\chi(G) = 2$ are called *bipartite*.

If G has a subgraph isomorphic to K_m, the complete graph on m vertices, then $\chi(G) \geq m$ because it requires m colors to color even that subgraph.

1. Give an example of a graph which does not contain a subgraph isomorphic to K_3 but with $\chi(G) \geq 3$.
2. Give an example of a graph which does not contain a subgraph isomorphic to K_4 but with $\chi(G) \geq 4$.
3. **Challenge**: for each $m \geq 3$, give an example of a graph G which does not contain a subgraph isomorphic to K_m but with $\chi(G) \geq m$.

Remark: In fact, much more is true. The *girth* of a graph G is the size of the smallest cycle in G. A classic result of Erdős (that is beyond what we'll prove in this class) says that for any g, m, there exists a graph G with girth $\geq g$ and $\chi(G) \geq m$.

Let $\Delta(G)$ denote the maximum degree of G. We saw a simple proof by induction that $\chi(G) \leq \Delta(G) + 1$.

4. Show that the bound just mentioned is sharp: for each $d \geq 2$, give an example of a graph with $\Delta(G) = d$ and $\chi(G) = d + 1$. How many examples can you think of?

Remark: *Brooks' theorem* says that the only G with $\chi(G) = \Delta(G) + 1$ are the "obvious" examples.

5. Let G be a bipartite graph on n vertices. How big can $\Delta(G)$ be?
6. For each $d \geq 1$, give an example of a bipartite graph G for which the *minimum degree* of G is d.