Math 4707: Intro to combinatorics and graph theory
Spring 2021, Instructor: Sam Hopkins
Final exam- Due Wednesday, May 5th

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are not allowed to to interact with anyone (including online forums) except for me, the instructor. As always, in order to earn points you need to carefully *explain your answer*.

1. (20 points total; 10 points each part)
   (a) How many paths are there in the plane $\mathbb{R}^2$ going from $(0, 0)$ to $(50, 100)$ taking unit steps in either the north or east direction at each step?
   (b) How many such paths are there which avoid passing through any of the 3 "bad" points
   $$(10, 11),\ (20, 42),\ (30, 85)?$$

2. (20 points) Let t_n denote the number of trees on n vertices, considered up to isomorphism (i.e., t_n is the number of "unlabeled trees" on n vertices). For example, the first several values of t_n are
   $$t_1 = 1,\ t_2 = 1,\ t_3 = 1,\ t_4 = 2,\ t_5 = 3,\ t_6 = 6, \ldots$$
   Prove that $t_n \leq \binom{2(n-1)}{n-1}$.
   Hint: Did we talk about the problem of counting unlabeled trees somewhere in the textbook and/or lectures?

3. (20 points total) Fix integers m, n with $1 \leq m \leq n$. In this problem we consider simple bipartite graphs G with bipartitions (X, Y), where $\#X = m$ and $\#Y = n$.
   (a) (5 points) Show that there exists such a G with $(m-1)n$ edges for which there is no matching M in G containing m edges.
   (b) (15 points) Show that for every such G with at least $(m-1)n+1$ edges, there must be a matching M in G containing m edges.

4. (20 points) Your friend hands you a convex polyhedron in $\mathbb{R}^3$ which has triangular and hexagonal faces (and no other kinds of faces), and for which every vertex belongs to three edges. How many triangular faces must this polyhedron have?

   Hint: find various equations that relate the numbers v, e, f, t, h of vertices, edges, faces, triangular faces, hexagonal faces, respectively.

5. (20 points; 10 points each part) Let G be a graph. Recall that the chromatic polynomial $\chi(G, k)$ is the polynomial in k which counts the number of ways to properly k-color the vertices of G.

   (a) Let T be a tree on n vertices. Compute $\chi(T, k)$.
   Hint: The answer only depends on n, not which particular tree on n vertices T is.

   (b) Let C_n be the cycle graph on n vertices. Compute $\chi(C_n, k)$.

   Caution: this question is asking about the *chromatic polynomial* of the graph, **not** the chromatic number.