

## Generating functions, Math 4707, Spring 2021

Let  $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, \dots$  with  $F_n = F_{n-1} + F_{n-2}$  be the Fibonacci number sequence. We saw that the generating function  $F(x) := \sum_{n=0}^{\infty} F_n x^n$  for the Fibonacci numbers is  $F(x) = \frac{x}{1-x-x^2}$  and, via partial fraction decomposition,  $F(x) = \frac{1/\sqrt{5}}{1-x(1+\sqrt{5})/2} - \frac{1/\sqrt{5}}{1-x(1-\sqrt{5})/2}$ . Using the geometric series  $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ , we got  $F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$  and thus  $F_n \approx \frac{1}{\sqrt{5}} \phi^n$  where  $\phi = \left(\frac{1+\sqrt{5}}{2}\right) \approx 1.618\dots$  is the golden ratio.

1. Consider the sequence  $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 3, J_4 = 5, \dots$  with  $J_n = J_{n-1} + 2J_{n-2}$ . Form the generating function  $J(x) := \sum_{n=0}^{\infty} J_n x^n$ . Show that  $J(x) = \frac{x}{1-x-2x^2}$ .
2. Use partial fractions to show that  $J(x) = \frac{1/3}{1-2x} - \frac{1/3}{1+x}$ .
3. Conclude that  $J_n = \frac{2^n - (-1)^n}{3}$ , so that  $J_n \approx \frac{1}{3} 2^n$ .

For  $n \in \mathbb{N}$ , the generating function for the binomial coefficients  $\binom{n}{k}$  is  $\sum_{k=0}^n \binom{n}{k} x^k = (1+x)^n$  (by the binomial theorem). Generalizing this, for any real number  $n \in \mathbb{R}$ , define  $\binom{n}{k} := \frac{n(n-1)(n-2)\dots(n-(k-1))}{k!}$ . The *generalized binomial theorem* says that for any  $n \in \mathbb{R}$  we have  $\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$ .

4. Show that  $(1+x)^{-4} = \sum_{k=0}^{\infty} (-1)^k \binom{4+k-1}{k} x^k$  and thus that  $\left(\frac{1}{1-x}\right)^4 = \sum_{k=0}^{\infty} \binom{4+k-1}{k} x^k$ .
5. Explain how the previous result relates to the “giving pennies to kids” / “choosing bagels” problem (hint: this case = 4 flavors of bagels).
6. Show that  $(1+x)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \dots (2k-1)}{2^k k!} x^k$  and thus that  $\frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$  is the *central binomial coefficient* generating function.
7. Use the previous result to show that for all  $k \in \mathbb{N}$ ,  $4^k = \sum_{j=0}^k \binom{2j}{j} \binom{2(k-j)}{k-j}$ .