Math 4707: More graph basics + trees

2/22

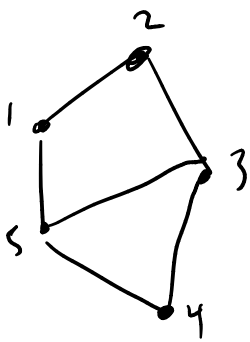
Ch's 7+8 of LPV

Reminder: • Midterm #1 due on Wed., 2/24

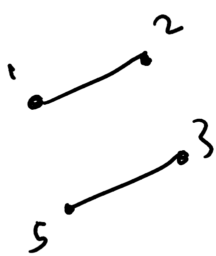
More graph basics

Last time we started graph theory. Let's continue with some basic notions + important families of graphs. For simplicity lets restrict to simple graphs today (no multiedges or loops) so that a graph G is a pair $G=(V,E)$ with $E \subseteq \{S \subseteq V : \#S = 2\}$ (edges are pairs of vertices).

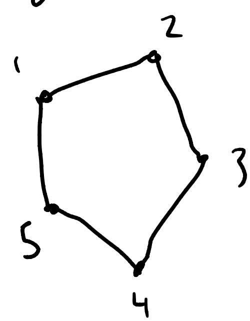
A subgraph of G is any graph obtained by deleting vertices and edges:



G



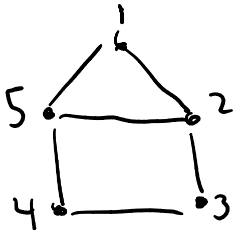
one subgraph



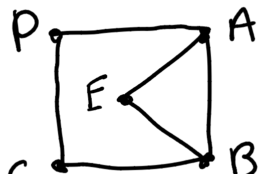
another subgraph

If $G=(V,E)$ and $G'=(V',E')$ are two graphs, then an isomorphism $f: G \to G'$ is a bijection $f: V \to V'$ on vertices such that

$$\{u,v\} \in E \iff \{f(u), f(v)\} \in E'.$$



G



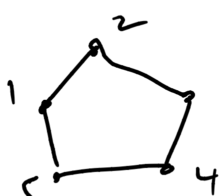
G'

$1 \longrightarrow E$
$2 \longrightarrow B$
$3 \longrightarrow C$
$4 \longrightarrow D$
$5 \longrightarrow A$

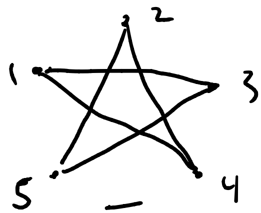
isomorphism

If G and G' are isomorphic, it really means they're the 'same' graph but w/ their vertices relabeled.

If G is a graph, then its complement $\overline{G}$ is the graph w/ the same vertices and w/ $\{u,v\}$ an edge of $\overline{G} \iff \{u,v\}$ not edge of G.



G

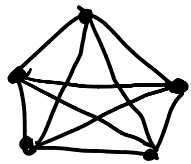


$\overline{G}$

Families of graphs

Now let's define some important graphs.

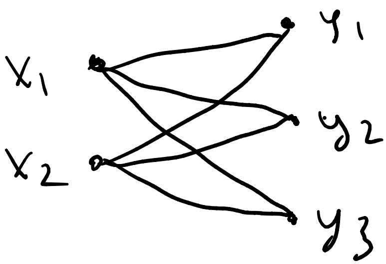
The complete graph K_n on n vertices has all pairs of vertices as edges:



K_5

The complement $\overline{K_n}$ is the empty graph w/ no edges.

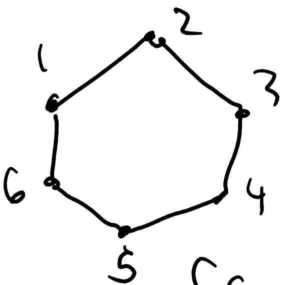
A related construction is the complete bipartite graph $K_{m,n}$ which has $n+m$ vertices $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_n$ and has edges $\{x_i, y_j\}$ $\forall i, j$, but no edges between the x's or the y's:



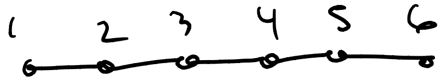
$K_{2,3}$

We'll explain the name 'bipartite' later...

The cycle graph C_n and the path graph P_n look like what you'd expect:

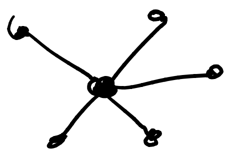


C_6



P_6

Sometimes $K_{1,n}$ is called a star:



$K_{1,5}$

Trees

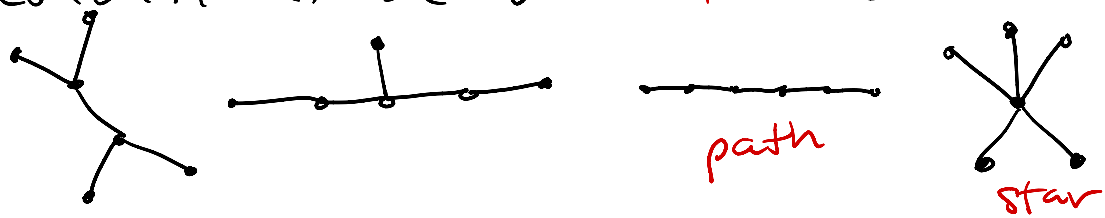
Last class we studied walking problems. Central to these problems was the notion of connectedness. It makes sense to be interested in the 'minimally connected' graphs. We will call these graphs trees...

Thm Let G be a graph. TFAE:

1) G is minimally connected, i.e., G is connected but the removal of any edge would disconnect G.

2) G is connected and contains no cycles.

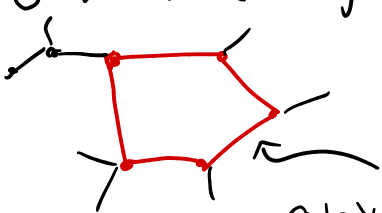
A graph satisfying either of these equiv. conditions is called a tree. Some trees:

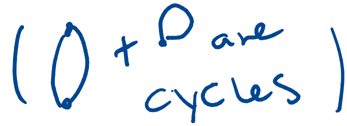


Pf of thm: Let G be a connected graph. We need to show:

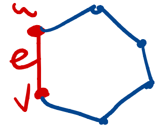
G has an edge we can remove + stay connected $\iff$ G has a cycle.

[$\Leftarrow$] Suppose G has a cycle:





Then we can remove any edge of the cycle without changing connectivity of graph.

[$\Rightarrow$] Suppose G has an edge $e=\{u,v\}$ we can remove + stay connected: 

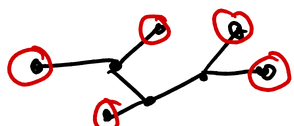
Then there has to be another path from u to v not using e, which forms a cycle with e. $\boxtimes$

Feels like:

- if G has too few edges, it can't be connected
- if G has too many edges, it will have a cycle

So trees are "goldilocks graphs" that have just the right # of edges. In fact:

Thm A tree with n vertices has $n-1$ edges.

In order to prove this theorem, we need a lemma. A leaf of a tree is a vertex of degree $= 1$. $\hookrightarrow$ 

Lemma Any tree (w ≥ 2 vertices) has a leaf.

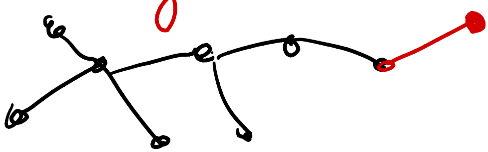
Remark: Can show that actually there must be at least two leaves.

Pf. of lem: Start at any vertex of our tree T and keep walking to new vertices along edges we haven't used:

$$v_0 \to v_1 \to \cdots \to v_k$$

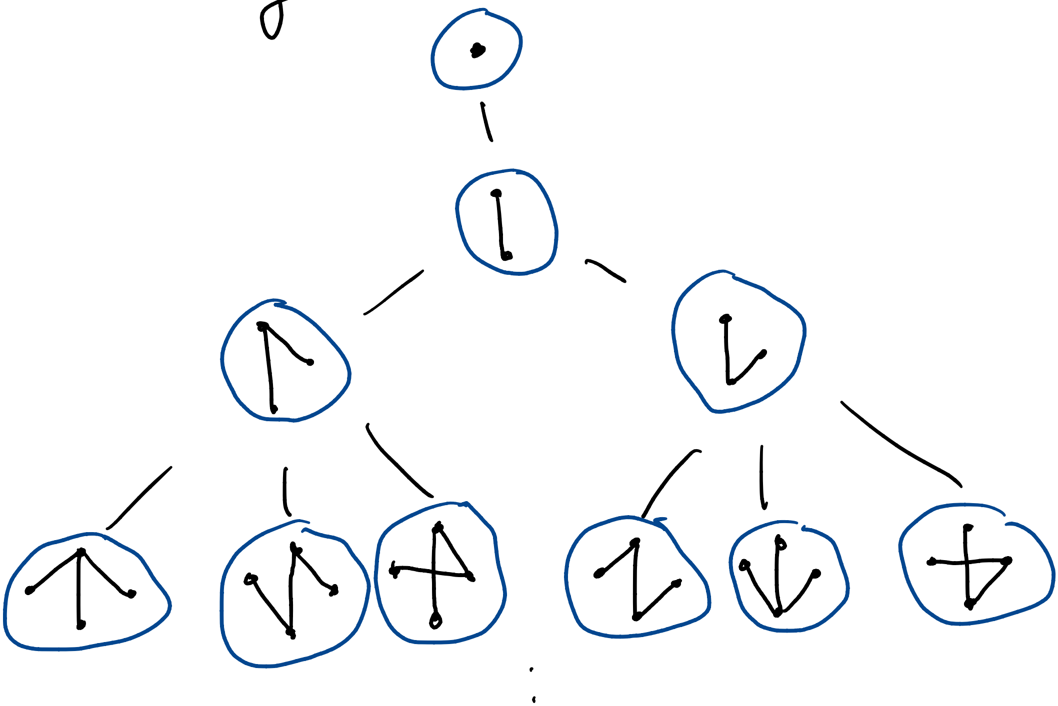
We don't have a cycle, so can never revisit a vertex. Eventually we get stuck: at a leaf. $\square$

Pf of thm: By lemma any tree on n vertices can be obtained from a tree on $n-1$ vertices by appending a leaf: (Think about this)



Thus, the theorem follows by induction, with the base case being tree w/ 1 vertex and zero edges: $\bullet$ $\square$

What we really just showed was that any tree can be obtained via the tree-growing procedure which starts w/ one vertex graph and repeatedly appends a new vertex connected by an edge to one of the existing vertices:



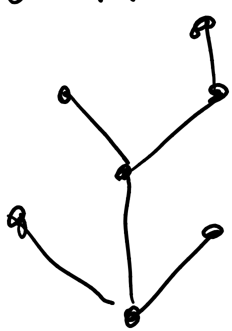
W/ tree-growing procedure, can prove many facts about trees, like:

Thm Let G be a graph on n vertices. Then any 2 of these implies the 3rd:

- G is connected.
- G has no cycles.
- G has $n-1$ edges.

Why are these called "trees"? Arboreal terminology makes most sense for rooted trees: a rooted tree is a tree where we've chosen a special root vertex, which we draw at the top, w/ other vertices branching down from it:

The picture makes most sense if we draw it upside-down:

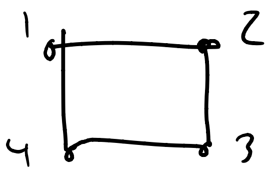


But traditional to draw it with root at top and use family tree terminology:

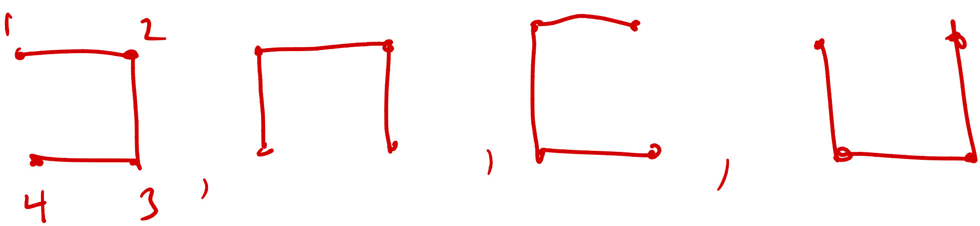
parent

child child child

Let G be a graph. A spanning tree of G is a subgraph that's a tree containing all the vertices of G.

e.g. $G =$ 

spanning trees:



Prop. G has a spanning tree $\iff$ G is connected.

Pf: ???

Now let's take a 5 min. break and when we come back we'll work on the free worksheet in breakout groups.