Math 4707: Directed graphs + tournaments

3/3

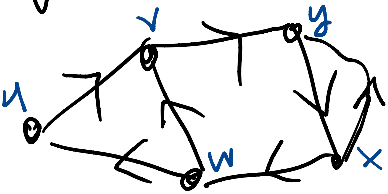
Not in LPV

Reminders: • Still working on HW#2 grading...

• HW#3 posted, due next Wed. 3/10.

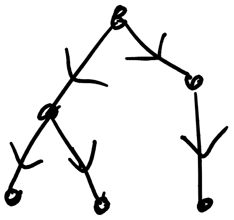
We'll continue today with some 'supplemental material' about graph theory.

Today we will discuss a new variant of graphs. A directed graph (or digraph) G is like a usual 'undirected' graph, except that each edge comes with an orientation, either 'u to v' or 'v to u'. We draw directed graphs using arrows:



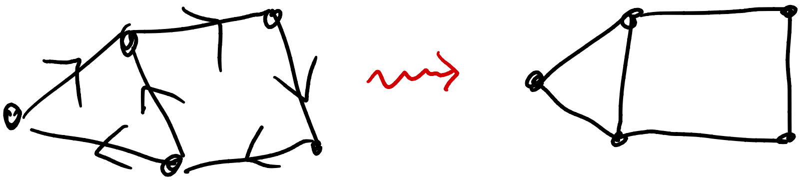
Formally, we represent the edges of a digraph using ordered pairs $e=(u,v)$.

Digraphs are useful for modeling non-symmetric relations. For example, we already saw this with rooted trees:



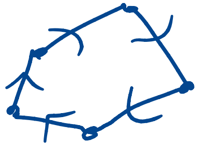
Here the relation could be 'is father of.'

Every directed graph has an underlying undirected graph obtained by forgetting orientations:



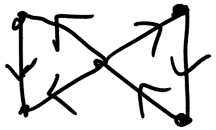
We can talk about walks, paths, cycles, etc. in the underlying undirected graph, but makes more sense to study directed walks, directed paths, etc., which are those that only traverse edges in the direction of their orientation:

directed path

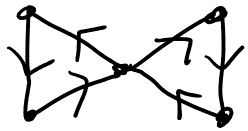


directed cycle

There are correspondingly two notions of connectivity for digraphs: G is **(weakly) connected** if its underlying undirected graph is connected, and is **strongly connected** if any two vertices are joined by a directed path in both directions:



(weakly) connected



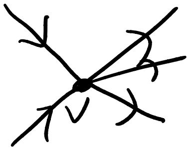
strongly conneted

Finally, we have two kinds of degrees of vertices of directed graphs:

indegree(v) = # arrows pointing into v

outdegree(v) = # arrows pointing out of v

e.g.

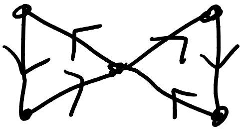


indegree = 2, outdegree = 3

Many of the results for undirected graphs we discussed have directed analogs

Thm For connected digraph G, $\exists$ a directed Eulerian circuit iff indegree(v) = outdegree(v) $\forall v$.

pf: Basically identical to undirected ... $\boxtimes$

e.g.  'one-way bridges'

There's a directed version of the adjacency matrix $A_G = (a_{ij})$:

$$a_{ij} = \begin{cases} 1 & \text{if } i \to j \\ 0 & \text{otherwise} \end{cases}$$

Note A_G is no longer symmetric!

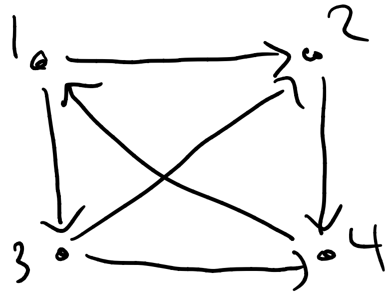
But $A_G^\ell[i,j]$ still counts # (directed) walks of length ℓ from i to j.

There's even a directed version of the Matrix-Tree Thm for counting rooted spanning trees.

Tournaments

An important class of directed graphs are tournaments: those whose underlying undirected graph is a complete graph:

e.g.



They're called tournaments b/c we can think of them as encoding results of a round-robin tournament where every team plays every other team once, and $i \to j$ means i beat j.

Given a tournament, we might want to rank the teams at the end:

t_1 ← 1st place, t_2, t_3, . . . , t_n ← last place

Ideally, each team would beat every team ranked lower than it (no upsets). Call the ranking a perfect ranking in this case. Only very special tournaments have perfect rankings.

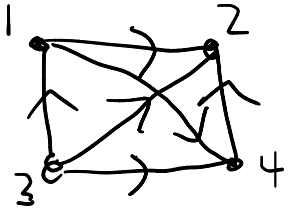
Say tournament T is transitive if whenever $i \to j$ and $j \to k$, then also $i \to k$.

Say T is acyclic if it has no directed cycles.

Thm For tournament T, TFAE:

- T has perfect ranking,
- T is transitive,
- T is acyclic

e.g.



✓

Need a lemma about all tournaments first:

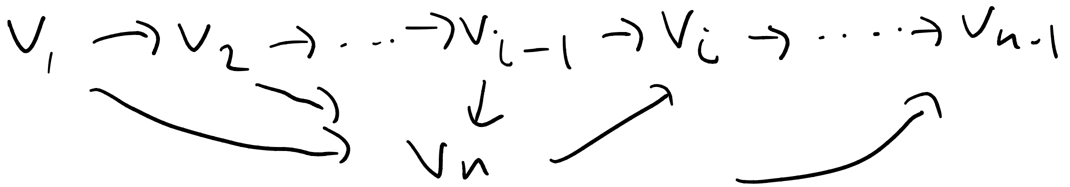
Lemma Every tournament has a directed Hamiltonian path.

Pf: By induction on # vertices. Base case ✓.

Assume by induction have path

$$v_1 \to v_2 \to v_3 \to \cdots \to v_{n-1}$$

and are adding vertex v_n. If $v_n \to v_1$, can add at beginning. And if $v_{n-1} \to v_n$, can add v_n at end. So assume that $v_1 \to v_n$ and $v_n \to v_{n-1}$:



Let v_i be smallest i s.t. $v_n \to v_i$.

Then $v_1 \to v_2 \to \cdots \to v_{i-1} \to v_n \to v_i \to \cdots \to v_{n-1}$ is our desired path. $\square$

Rmk: Can show # ham. paths is odd!

Pf of thm: 1 ⇒ 2: if perfect ranking is $v_1, v_2, \ldots, v_n$, then arrows go from lower index to higher, so transitive.

2 ⇒ 3: Suppose we had a cycle. Let $v_1 \to v_2 \to \cdots \to v_k \to v_1$ be cycle of min. length. By transitivity, $v_{k-1} \to v_1$, so $v_1 \to \cdots \to v_{k-1} \to v_1$ is a shorter cycle, a contradiction.

3 ⇒ 1: By lemma, have a ham. path $v_1 \to v_2 \to \cdots \to v_n$. Let this be ranking. Any upset would yield a cycle. ☒

Q: How many transitive tournaments on n vertices are there?

Q: How many total tournaments on n vertices are there?

Note: Can show ∃ unique ham. path ⟺ T is transitive.

Ranking tournaments

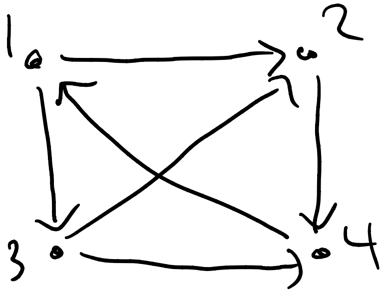
If we have a transitive tournament, we know how to rank the teams. But what about for non-transative T?

Common to define a score vector

$S_1 = (S_1(1), S_1(2), \ldots, S_1(n))$

w/ $S_1(i) = \#$ teams i beat.

e.g.



$S_1 = (2,1,2,1)$

But we might want to give more credit for beating batter teams, So define secondary score vector

$S_2 = (S_2(1), S_2(2), \ldots, S_2(n))$

w/ $S_2(i) = \Sigma S_1(j)$ for teams j that i beat

e.g. $S_2 = (3,1,2,2)$ for above T

And we can keep going, defining

$$S_k = (S_k(1), S_k(2), \dots, S_k(n))$$

w/ $S_k(i) = \sum_{i \to j} S_{k-1}(j)$.

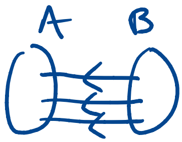
e.g. $S_3 = (3, 2, 3, 3)$

$S_4 = (5, 3, 5, 3)$

$S_5 = (8, 3, 6, 5)$ ...

We might wonder if $\lim_{k \to \infty} S_k$ exists?

Thm (Kendall-Wei method) no



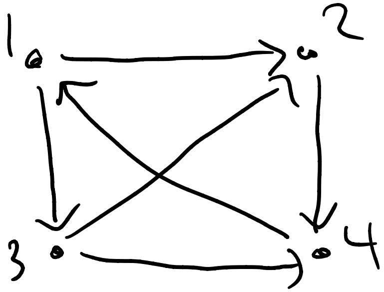
If T is irreducible then its adjacency matrix A_G has a largest real eigenvalue λ_1 w/ 1-dim'l eigenspace,

and $\lim_{k \to \infty} \frac{S_k}{\sum_i S_k(i)}$ is a λ_1-eigenvector.

↗

normalize so sum of entries = 1

e.g.



$$A_G = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

Wolframalpha tells me that

$\lambda_1 \approx 1,395 \ldots$

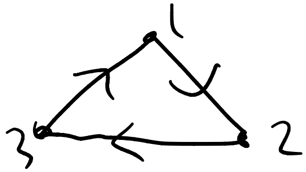
w/ eigenvector

$(.321\ldots, .165\ldots, .283\ldots, .230\ldots)$

So Kendall-Wei method says we rank:

1, 3, 4, 2

e.g.



$$A_G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

largest eigenvalue: $\lambda_1 = 1$

w/ eigenvector $(1/3, 1/3, 1/3)$

No way to rank teams!