Math 4707: Stable Marriages

3/22
Not in LPV

Reminders:
- HW #4 due this Wed. 3/24.
- Midterm #2 will be posted by this Wed., due next Wed. 3/31.
- Working on grading HW #3...

For the last several classes we've been discussing the problem of finding a perfect/maximal matching in a bipartite graph. Today we will discuss a different "matching" problem: the so-called stable marriages problem.

Consider the following scenario: there are 4 doctors graduating from medical school who want work at hospitals; and there are 4 hospitals which want to hire new doctors. What is the "optimal" way of matching the new doctors to hospitals? Must take into account the preferences of both the doctors + hospitals, which we record in 2 tables like this:

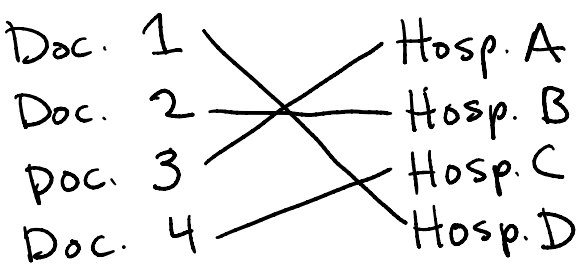
	Hosp. Pref.'s
Doctor 1	A, C, D, B
Doctor 2	A, D, B, C
Doctor 3	B, A, D, C
Doctor 4	B, A, C, D

	Doctor Pref.'s
Hosp. A	3, 2, 1, 4
Hosp. B	4, 2, 3, 1
Hosp. C	1, 3, 2, 4
Hosp. D	1, 2, 3, 4

e.g., Doc. 1 likes Hosp. A most, Hosp. B least.
Hosp. A likes Doc. 3 most, Doc. 4 least.

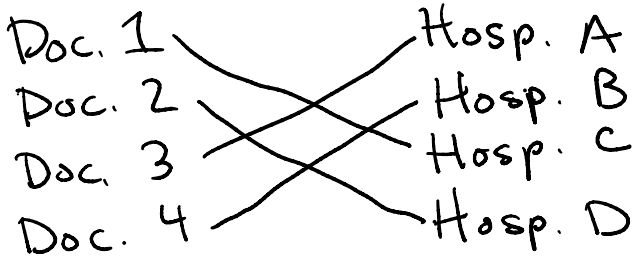
So... given this info about Doc./Hosp. preferences, how can we find 'optimal' or at least 'good' matching?

Clearly cannot just assign every Doc. to their top Hosp., or every Hosp. its top Doc. But can at least hope to find a stable matching, in following sense. Consider this assignment:



Something is very bad about this matching: Doc. 1 would prefer to be at Hosp. C rather than

D where she is currently assigned; AND Hosp. C would rather have Doc. 1 than 4 who is assigned there. Thus Doc. 1 + Hosp. C have incentive to break off agreement and be with each other: they are an unstable pair in this matching. Call a matching stable if it has no unstable pairs:



Can check that this matching is stable.

The "stable marriages" problem is problem of finding a stable matching given preference tables. We will show that a stable matching always exists, although not nec. unique:

e.g.

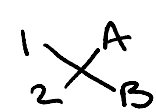
Doc. 1	A, B
Doc. 2	B, A

Hosp. A	2, 1
Hosp. B	1, 2

BOTH STABLE:

1 — A
2 — B

and



To find a stable matching, will use the Gale-Shapley 'deferred acceptance' algorithm, which works as follows:

- In 1st round, each Doc. 'proposes' to their top Hosp. Then each Hosp. says 'maybe' to the best proposal they got, and 'no' to all others. Hosp.'s are only tentatively matched to maybes.
- In 2nd + further rounds, each Doc who is not tentatively matched to a Hosp. 'proposes' to their top choice among Hosp.'s that haven't rejected them. Each Hosp. sends a 'maybe' to its best offer on this and previous rounds — and 'no' to all others. Thus, a Hosp. may break a tentative match from a previous round.
- The process continues until all Doc.'s are tentatively matched to Hosp.'s, at which moment the tentative matching becomes permanent.

e.g., with our running 4 Doc./Hosp. example:

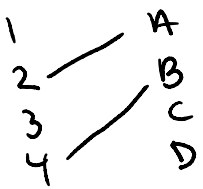
Round 1 Doc. 1 proposes to Hosp. A ← her top choice
2 proposes to A
3 proposes to B
4 proposes to B

1	A, C, D, B
2	A, D, B, C
3	B, A, D, C
4	B, A, C, D

A	3, 2, 1, 4
B	4, 2, 3, 1
C	1, 3, 2, 4
D	1, 2, 3, 4

A says no to 1 + maybe to 2.
B says ~~no~~ no to 3 + maybe to 4.

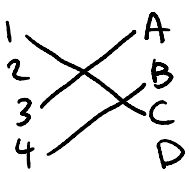
tentative matching:



Round 2 1 proposes to C ← her 2nd choice + 3 proposes to A

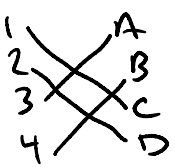
A says no to 2 (breaking match!) + maybe to 3
C says maybe to 1.

tentative matching:



Round 3 2 proposes to D + D says maybe to 2.

tentative matching:



← Same stable matching from before

Everyone matched, so we make this permanent!

Thm The G-S alg. yields a stable matching.

Pf: • Alg. terminates b/c Doc.'s will propose to all Hosp.'s, if necessary, and once a Hosp. has been proposed to it will have a tentative match from then on.

• Also, not possible that Doc. 1 + Hosp. A, say, prefer one another to current matches: 1 would have proposed to A before current match + A would only say no for a better offer. ▨

Can also run algorithm w/ Hosp.'s as proposers and Doc.'s as acceptors. On the worksheet you will explore this, and you will see that it is better to be a proposer!

[Aside about how this alg. is actually used for residency match, but in 'wrong direction' for decades. Also: Nobel Prize in Economics for Shapley + Roth...]

Now let's take a 5 min. break

and when we come back,
practice using the G-S alg.

on today's worksheet
in breakout groups.