Math 4707: Pick's Theorem

3/29

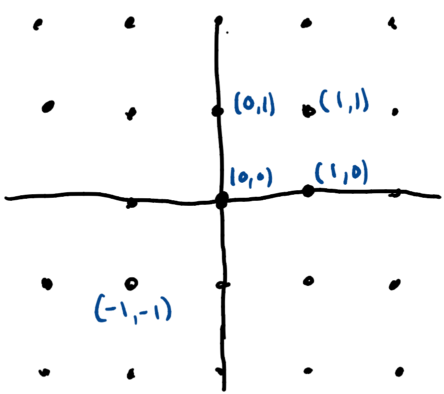
Not in LPV

Reminders: • HW #4 has been graded (that was fast!).

• Midterm #2 due this Wednesday, 3/31.

Last class we transitioned into a discussion of geometry and combinatorics. In particular, we discussed some combinatorial problems in the two-dimensional plane, with a particular focus on polygons. Today we will study lattice polygons and present a beautiful theorem of Pick which tightly links the continuous and discrete.

By lattice we mean $\mathbb{Z}^2$, the two-dimensional grid consisting of points in the plane with integer coordinates. It looks like this:

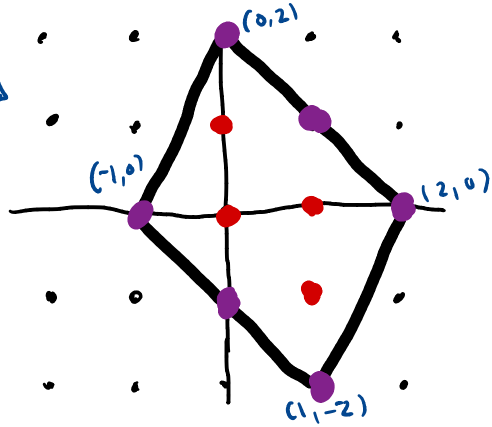


A lattice polygon is a polygon whose vertices are in $\mathbb{Z}^2$:

this quadrilateral $\rightarrow$

P has vertices: $(0,2), (-1,0), (1,-2), (2,0)$

and so is a lattice polygon



= 'internal' lattice pts. of P

= 'boundary' lattice pts. of P

A natural thing to do is count the lattice points of a lattice polygon P, i.e. pts of $\mathbb{Z}^2$ contained in P. Two kinds:

- 'internal' lattice pts of P: points in $\mathbb{Z}^2$ strictly inside P
- boundary lattice pts of P: pts in $\mathbb{Z}^2$ along boundary of P.

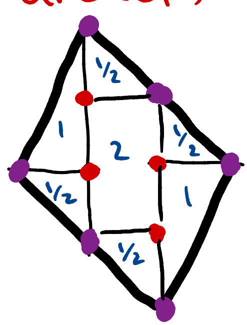
Let us use notation:

- $i(P)$ = # of internal lattice pts of P,
- $b(P)$ = # of boundary lattice pts of P.

In above example we have $i(P) = 4$ and $b(P) = 6$

We also have that area(P) = 6, as can be seen by:

$6 = 2 + 1(2) + \frac{1}{2}(4)$



$\leftarrow$ inside each rectangle or right triangle we write its area

Now let's do a funny equation based on these #'s:

$$\begin{aligned} \text{area}(P) &= 6 \\ &= 4 + \tfrac{1}{2}(6) - 1 \\ &= i(P) + \tfrac{1}{2} b(P) - 1 \end{aligned}$$

Pick's theorem says this funny equation is always true:

<u>Thm</u> (Pick's theorem) For any lattice polygon P,

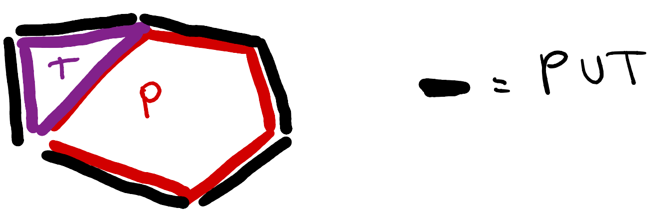
$$\text{area}(P) = i(P) + \frac{b(P)}{2} - 1.$$

Beauty of thm = connects continuous (area) + discrete (lattice pts).

In the rest of the lecture we'll sketch a proof of Pick's thm. The proof is <u>inductive</u>, based on decompositions of polygons.

<u>Lemma</u> Let P be a <u>lattice polygon</u> and T a <u>lattice triangle</u> such that P and T intersect along a <u>common side</u>:

<u>e.g.</u>



Let $P \cup T$ denote lattice polygon that's <u>union</u> of P and T.

Then $\boxed{\text{area}(P\cup T) = \text{area}(P) + \text{area}(T)}$, and

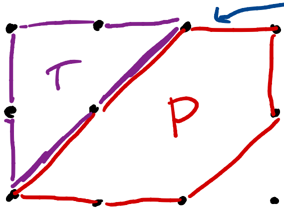
$$\boxed{i(P\cup T) + \frac{b(P\cup T)}{2} - 1 = \left(i(P) + \frac{b(P)}{2} - 1\right) + \left(i(T) + \frac{b(T)}{2} - 1\right)}.$$

In particular, if Pick's Thm holds for any two of P, T, & $P \cup T$, then it holds for the 3rd as well!

Pf: The claim that areas are additive is basic geometry. Actually, it's essentially the definition of area. Now for the lattice pt counts, let

$$c := \#\text{ lattice pts on common side of } P \,\&\, T$$

e.g.



here $c = 3$, for 4 pts on this side

Then $i(P \cup T) = i(P) + i(T) + (c-2)$, b/c all pts on common side become internal, except for endpoints.

Similarly $b(P \cup T) = b(P) + b(T) \underbrace{-2c}_{\text{take away pts on common side}} \underbrace{+2}_{\text{add back in endpoints}}$

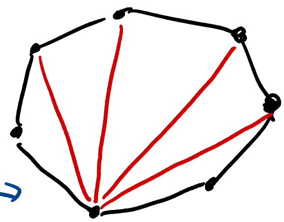
Thus,
$$i(P \cup T) + \frac{b(P \cup T)}{2} - 1 = i(P) + i(T) + c - 2 + \frac{b(P)}{2} + \frac{b(T)}{2} - c + 1 - 1$$
$$= \left(i(P) + \frac{b(P)}{2} - 1\right) + \left(i(T) + \frac{b(T)}{2} - 1\right). \quad \boxtimes$$

Lemma says that we can prove Pick's thm by building up or whittling down to lattice polygons we already know it for.

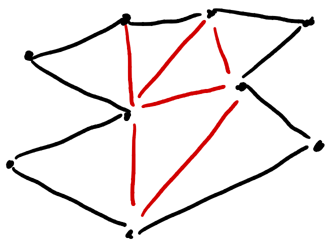
Lemma Every (lattice) polygon P has a triangulation, i.e., a dissection into (lattice) triangles, intersecting along common sides.

For convex polygons, easy to triangulate:

can pick any vertex and fan out triangles from it



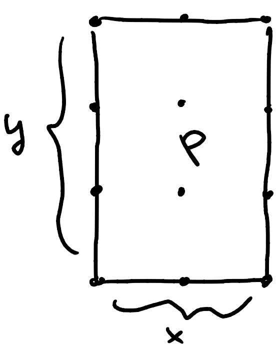
But even non-convex polygons have triangulations:



I'll leave proof for non-convex P as exercise for you!

These 2 lemmas reduce pf. of Pick's thm to case of lattice triangles, which we do in several steps:

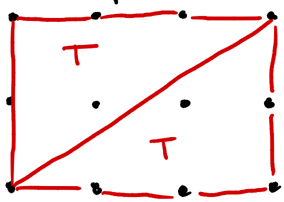
① Check Pick for rectangles w/ sides parallel to axes:



$$\text{area}(P) = (x-1)(y-1)$$
$$i(P) = (x-2)(y-2)$$
$$b(P) = 2x + 2y - 4$$ ← add 4 sides, subtract 4 vertices

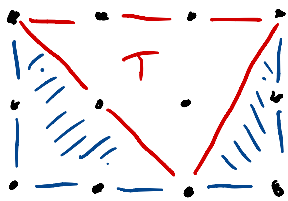
$$\Rightarrow \text{area}(P) = i(P) + \frac{b(P)}{2} - 1 \checkmark$$

② From rectangle case, deduce it for right triangles w/ 2 sides parallel to axes:



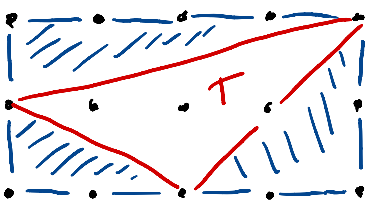
glue two copies of T together to form a rectangle + apply the dissection lemma

③ For lattice triangles w/ 1 side parallel to an axis, can use a dissection like this:

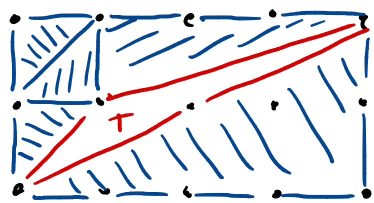


we already know Pick for blue triangles and big rectangle

④ For arbitrary lattice triangles, use dissection like:

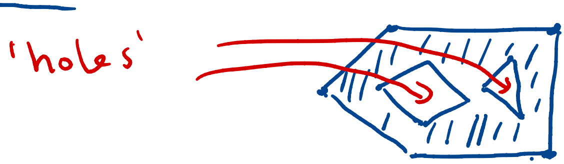


or



These steps complete pf of Pick's Thm. //

Remark: Can extend Pick to (lattice) polygons w/ holes:



= polygon w/ holes

Thm $\text{area}(P) = i(P) + \frac{b(P)}{2} - 1 + \#\text{holes}(P)$

Now let's take a 5 min. break,

and when we come back

do the Pick's Thm worksheet

in breakout groups!