Math 4707: Planar graphs + Euler's formula

3/31

Ch. 12 of LPV

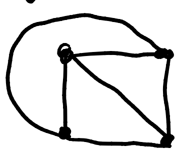
Reminder: • Midterm #2 is due today.

We continue discussing some interactions of planar geometry and combinatorics, specifically graph theory.

For convenience, today all graphs will be simple (no multi edges, loops).

DEF'N A planar graph is a graph which can be drawn in the plane s.t. edges do not cross (except at vertices).

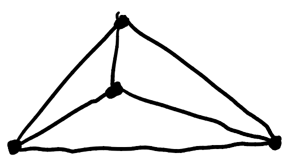
e.g. The complete graph K_4 is planar since



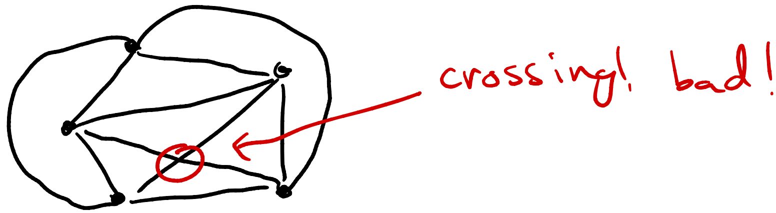
← no crossing edges

Notice that we are allowed to use 'bendy' edges; but in fact it's a theorem that if we can find a planar embedding w/ 'bendy' edges, we can find one w/ 'straight' edges:

e.g. $K_4 =$



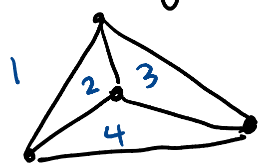
In a bit we will show that K_5 is not planar:



The main result about planar graphs we will discuss today is Euler's formula (see Ch. 12 of the text).

To present Euler's formula, we need the notion of faces of a planar embedding of a graph G:

e.g.



As we indicated, the planar embedding divides the plane into connected regions: 4 regions in the example.

These regions are called the faces of G. Note there is always one big "outer face" on the "outside" of G: the region labeled 1 in the example.

Rmk Can be helpful to think of planar graphs as maps, with the faces corresponding to different countries. We will study coloring maps in a future class.

Euler's formula is a relation between $f := \#\text{faces}(G)$, $e := \#\text{edges}(G)$, and $v := \#\text{vertices}(G)$.

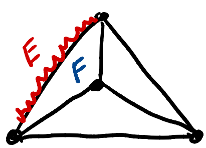
<u>Thm</u> (**Euler's formula**) For a connected planar graph G,

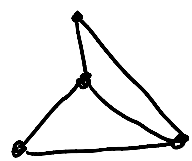
$$f + v = e + 2.$$

<u>e.g.</u> for K_4 above, $f = 4, v = 4, e = 6 \Rightarrow 4 + 4 = 6 + 2$ ✓

<u>Pf:</u> First suppose that G has some face besides the outer face. One of the faces must "border" the outer face (this should be 'geometrically obvious'). Choose such a face F and let E be an edge on border w/ outer face:

<u>e.g.</u>

$G =$ 

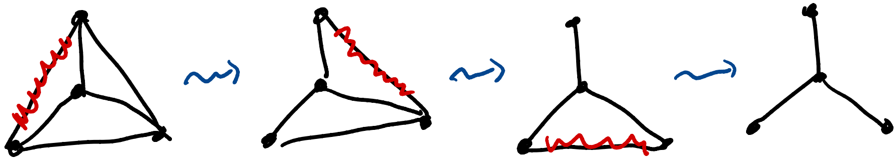
$\rightsquigarrow$ $G' =$ 

As shown above, form graph G' by deleting edge E. Observe that $e(G') = e(G) - 1$, and also $f(G') = f(G) - 1$ since F and the outer face 'merge together' in G'. So if we look at Euler's formula

$$f + v = e + 2,$$

we see it holds for $G \iff$ it holds for G' since LHS and RHS are one bigger for G compared to G'.

We thus repeatedly 'peel off' edges from G:

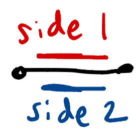


We stop peeling when the only face is the outer face. In this case, our graph G is connected but has no cycles (otherwise there would be a face), so G is a tree T! And we know that $e(T) = v(T) - 1$, while $f(T) = 1$ (the outer face), so Euler's formula holds for trees. $\blacksquare$

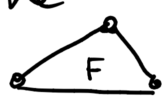
From Euler's formula we get a bound on #edges(G).

Cor Let G be a connected planar graph w/ $n > 2$ vertices. Then #edges$(G) \leq 3n - 6$.

Pf: Think of each edge as having two 'sides':



Each face of G takes at least 3 sides, b/c the smallest possible face is a triangle:



This means $\#\text{edges} = \frac{1}{2}\#\text{sides} \geq \frac{3}{2}\#\text{faces}$,

i.e., $\#\text{faces} \leq 2/3\ \#\text{edges}$.

From Euler's formula we get

$$\#\text{edges} + 2 = n + \#\text{faces} \leq n + 2/3\ \#\text{edges}$$

which we can rewrite as

$$\#\text{edges} \leq 3n - 6.$$

$\blacksquare$

Cor K_5 is not planar.

Pf: K_5 has $\binom{5}{2} = 10$ edges, and $n = 5$ vertices, but $3n - 6 = 15 - 6 = 9 < 10$, so by last corollary can't be planar. $\boxtimes$

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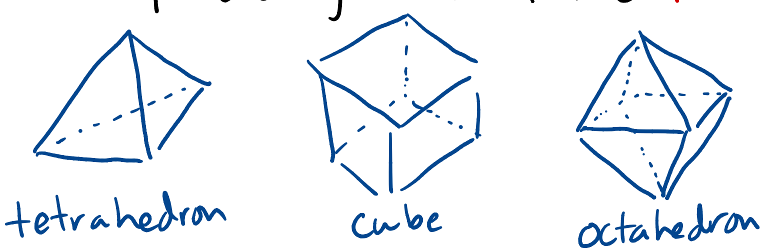
The context in which Euler's formula was first discovered (by Euler) was for face, edge, vertex #'s of convex polyhedra. Polyhedra are the 3D versions of polygons (convex polyhedra being analog of convex polygons). You are probably aware of the Platonic solids:



dodecahedron + icosahedron

Just like polygons have vertices and edges ('sides'), polyhedra have vertices, edges, and faces, where the faces are polygons.

Euler noticed that his formula

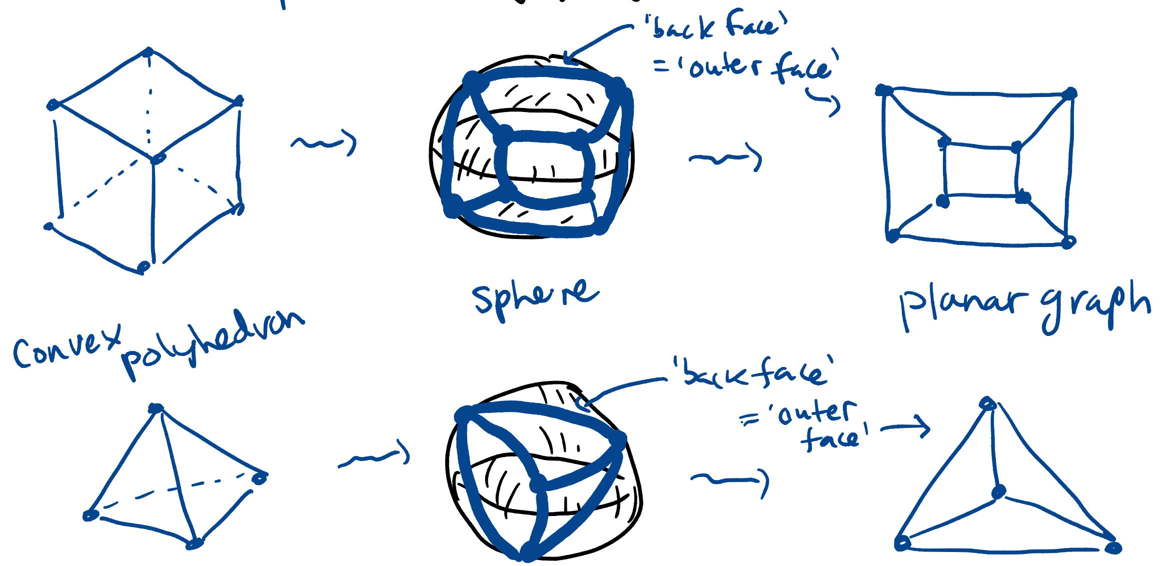
$$f + v = e + 2$$

holds for convex polyhedra.

e.g. $6 + 8 = 12 + 2$ for the cube

In fact, we can use Euler's formula for planar graphs to prove it for convex polyhedra.

Idea is to "blow up" polyhedron into a sphere, and then "expand" one of faces into outer face:



We'll discuss planar graphs more in future lectures. For today, there is no worksheet. Instead, focus on getting your Midterm #2 turned in today, and then have an excellent (and safe) Spring Break!