Math 4707: Coloring Maps and Planar Graphs

4/14

Ch. 13 of LPV

Reminder: • HW #5 due in 1 week on Wed., 4/21.

Last class we introduced coloring for graphs. Two classes ago we introduced planar graphs. Today we will compine these two topics.

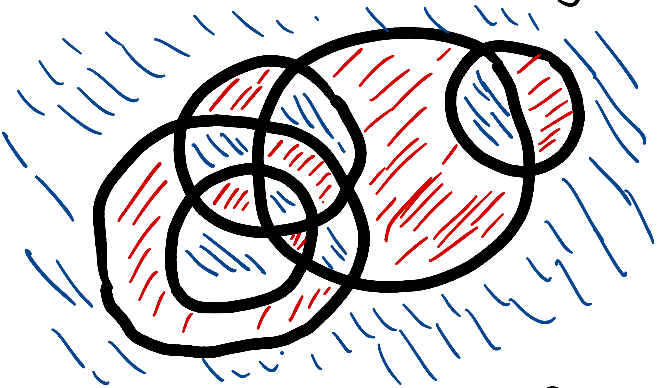
First we will discuss coloring maps. Imagine we have some division of the plane into regions:

As suggested by this picture, our goal is to color the regions so that adjacent regions are colored differently.

If we think of this picture as a map, then the regions are countries, and we want bordering countries to be colored differently.

Main question: how few colors do we need to color any map?

Some special kinds of maps are easy to color. For instance, if map is made by intersecting circles:



see book for proof. or exercise 13.4.8 on HW.

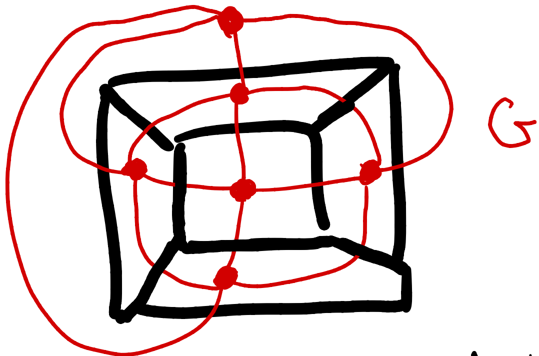
then can easily 2-color it. But definitely need more than 2 colors for most maps...

The Four Color Theorem, one of the most famous theorems in graph theory, says you only ever need four colors to color a map.

Proof of 4-color thm is very complicated and controversial for using computer help!

We won't prove the 4-color thm, but we will prove the Five Color Theorem, saying you can always color a map w/ 5 colors.

To get started, let's try to translate back into language of graph coloring from last class. For any map, construct a graph G:



The vertices of G correspond to the regions of the map, and we draw an edge between two vertices when corresponding regions are adjacent. Because the map was planar, G will be a planar graph.

(This construction is called the dual of a planar graph, we'll discuss it next class...).

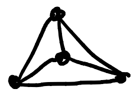
And a coloring of regions of map s.t. adjacent regions get different colors

$\Leftrightarrow$ a proper vertex-coloring G.

So we can reformulate 4-color thm as:

Thm (Four Color Theorem)

$\chi(G) \leq 4$ for any planar graph G.

Rmk Recall $K_4 =$  is planar, so certainly can't go lower than 4.

As mentioned, we'll instead prove $\chi(G) \leq 5$ for planar G. First, let's prove $\chi(G) \leq 6$, which should be even easier...

Lemma Every planar graph has a vertex of degree ≤ 5.

Pf: Let G be planar. Assume every vertex of G has degree ≥ 6. Then

$$\#\text{edges}(G) = \frac{1}{2} \sum_v \deg(v) \geq \frac{1}{2} 6n = 3n,$$

where $n = \#\text{vertices}(G)$.

But recall from Euler's formula we

deduced the bound

$$\#\text{edges}(G) \le 3n - 6 \quad (\text{if } n \ge 3),$$

a contradiction. $\boxtimes$

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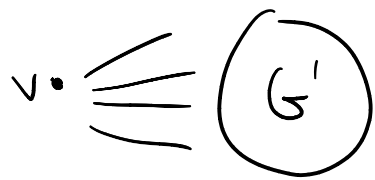
Recall last class we proved that if <u>every</u> vertex of a graph G has degree $\le d$, then $\chi(G) \le d+1$. We only know that <u>some</u> vertex of G has degree ≤ 5, so doesn't quite seem to apply. However, if you remember our <u>inductive</u> proof



it actually proves the following.

<u>Lemma</u> If every subgraph H of a graph G has some vertex of degree $\le d$, then $\chi(G) \le d+1$.

Since every subgraph of a planar graph is planar, we conclude:

Cor $\chi(G) \leq 6$ for all planar graphs G.

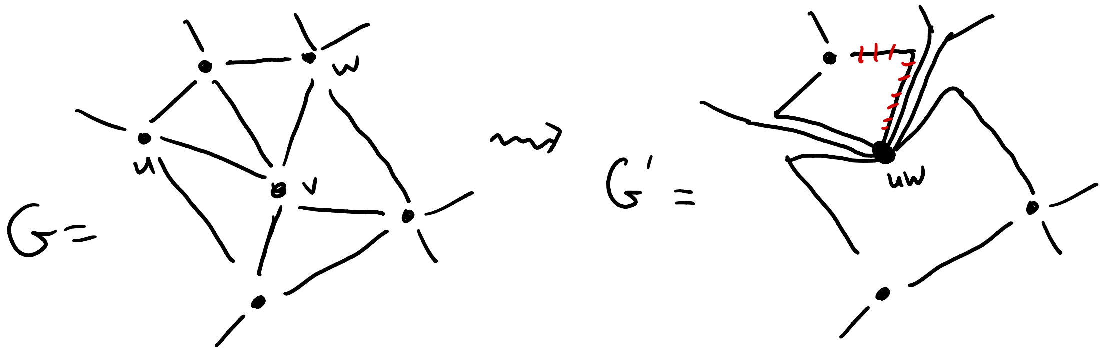
To prove the Five Color Theorem we can use the same kind of inductive argument, with a little bit more care.

Thm $\chi(G) \leq 5$ for any planar graph G.

Pf: Let v be vertex of G w/ $\deg(v) \leq 5$, which we know exists by the above lemma.
If $\deg(v) \leq 4$, then by induction can color $G - \{v\}$ with 5 colors, and since v's neighbors only use 4 colors, have at least one color left to color v.

So now assume that $\deg(v) = 5$. We claim that among the 5 neighbors of v, there are two vertices u and w which are not adjacent. Indeed, otherwise neighbors of v form a K_5, which we know is non-planar.

So v, u, w look like:



As shown above, form graph G' from G by:

- removing v (and all edges incident to v),
- "merging" u and w into one vertex uw,
- deleting any parallel edges formed (~~///~~).

We see there is "space" for u and w to merge since v was removed, so G' is planar. By induction, we can 5 color G'. Then we get a 5-coloring of G by:

valid since u, w not adjacent

- coloring u and w same color as uw in G',
- coloring v remaining color, since its neighbors use at most 4 colors (since u, w same color). ◻

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History of 4-color Theorem:

- ~1850 conjectured by F. Guthrie, trying to color map of counties of England.

- 1879: A. Kempe thought he proved 4-color theorem, but proof was flawed. Really just proved 5-color thm (similar to what we did above).

- 1976: Appel + Haken announced proof of 4-color thm. Idea is:
  - reduce to 1,834 possible counterexamples,
  - have powerful computer check that all of these graphs can really be 4-colored.

To date, no fully "human understandable" proof of 4-color theorem is known! //