Math 4707: Chromatic polynomial

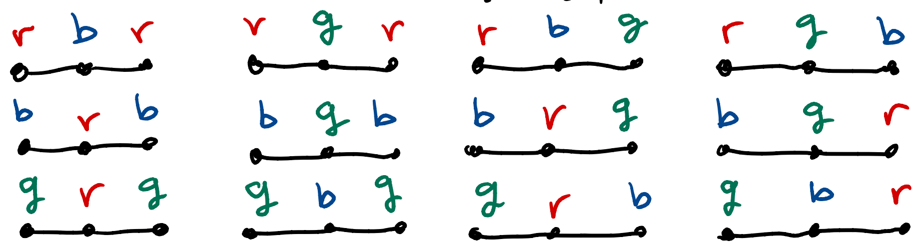
4/21
Not in LPV

Reminders: • HW #5 is due today!
• SRTs ongoing... please fill out by 5/3 (last day of class).

Today we will count the number of (proper) vertex colorings of a graph. This leads to the chromatic polynomial, an important graph invariant. This is the last topic from the course that you might be evaluated on (hint: there might be a Q about chromatic poly. on final!).

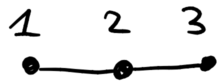
Let G be a (simple) graph. For a positive integer k, let $\chi(G,k)$ denote the # of proper vertex colorings of G using k colors.

For example, if $G = P_3$ is the path graph on 3 vertices, then $\chi(P_3,3) = 12$ since



are the ways to 3-color P_3. But instead of trying to compute the values of $\chi(G,k)$ for each individual k, let's be more systematic.

For instance, to compute $\chi(P_3,k)$ for any k, we can imagine coloring vertices 1, 2, 3 in order:

$P_3 =$ 

For 1 we can choose any of the k colors. For 2, we can choose any color except the one we used for 1, which is $(k-1)$ colors. Similarly, for 3 we can choose any color except the one for 2. So $\chi(P_3,k) = k\cdot(k-1)^2$.

This agrees w/ our above computation $\chi(P_3,3) = 3\cdot 2^2 = 12$.

Also note that $\chi(P_3,1) = 1\cdot 0^2 = 0$, which is correct since we cannot 1-color the vertices of P_3.

But the most important thing about the formula

$$\chi(P_3,k) = k\cdot(k-1)^2$$

is that it's a polynomial in k. This is something we'll show is true for all graphs G.

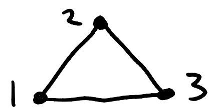
Thus $\chi(G,k)$ is called the **chromatic polynomial** of the graph G.

First let's see some more examples...

The exact same reasoning as for P_3 will show that for the path graph P_n on n vertices, have

$$\chi(P_n, k) = k \cdot (k-1)^{n-1}.$$

What about for the triangle K_3? Well again let's think of coloring the vertices 1, 2, 3 in order:

$K_3 =$ 

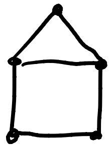
For 1 have k colors. For 2 have $(k-1)$ colors b/c have to be different from 1. For 3 have $(k-2)$ colors b/c have to be different from 1+2. So $\chi(K_3, k) = k(k-1)(k-2)$.

The same reasoning shows for complete graph K_n,

$$\chi(K_n, k) = k(k-1)(k-2)\cdots(k-(n-1)).$$

Another important case is when G is the empty graph on n vertices (i.e., n vertices + no edges) then $\chi(G, k) = k^n$ since we have n independent choices of k colors.

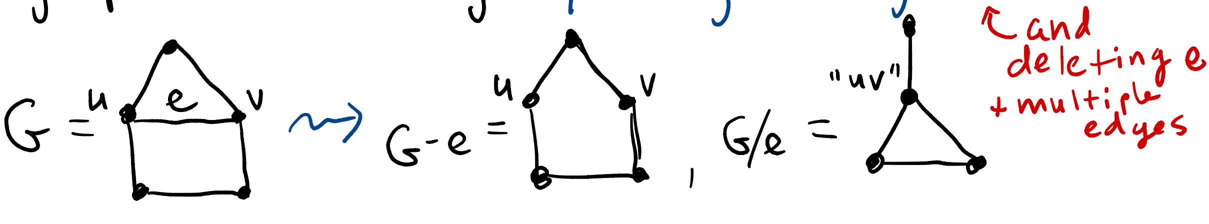
This kind of analysis works for some graph families, but if we considered

$G =$ 

it would be hard to compute $\chi(G,k)$ by thinking about coloring vertices one at a time.

To show $\chi(G,k)$ is a polynomial in k for any G, we need another strategy... we need to use deletion and contraction.

For any edge $e = \{u,v\}$ of our graph G, the deletion $G-e$ of e is the graph we obtain by removing e. The contraction G/e of e is the graph we obtain by "squeezing $u+v$ together":

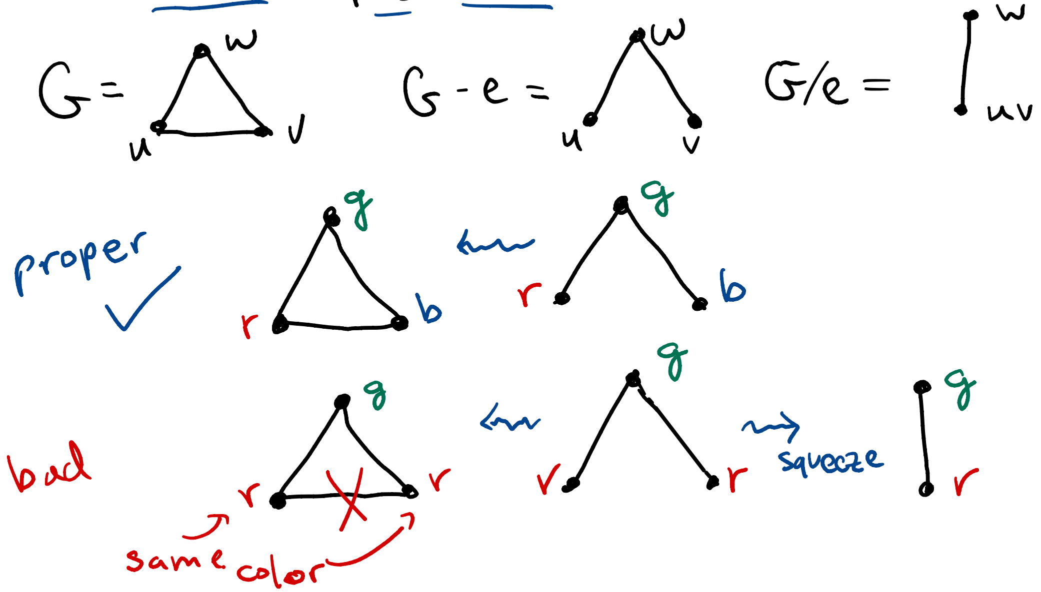


We can use deletion/contraction to give a recursive formula for the chromatic polynomial (where base case = empty graphs).

Thm (Deletion/contraction formula for $\chi(G,k)$)

For any edge e of G, $\chi(G,k) = \chi(G-e,k) - \chi(G/e,k)$.

Pf: Since $G-e$ differs from G only in that it lacks edge $e=\{u,v\}$, any proper coloring of $G-e$ will be a proper coloring of G, unless u and v are colored the same:

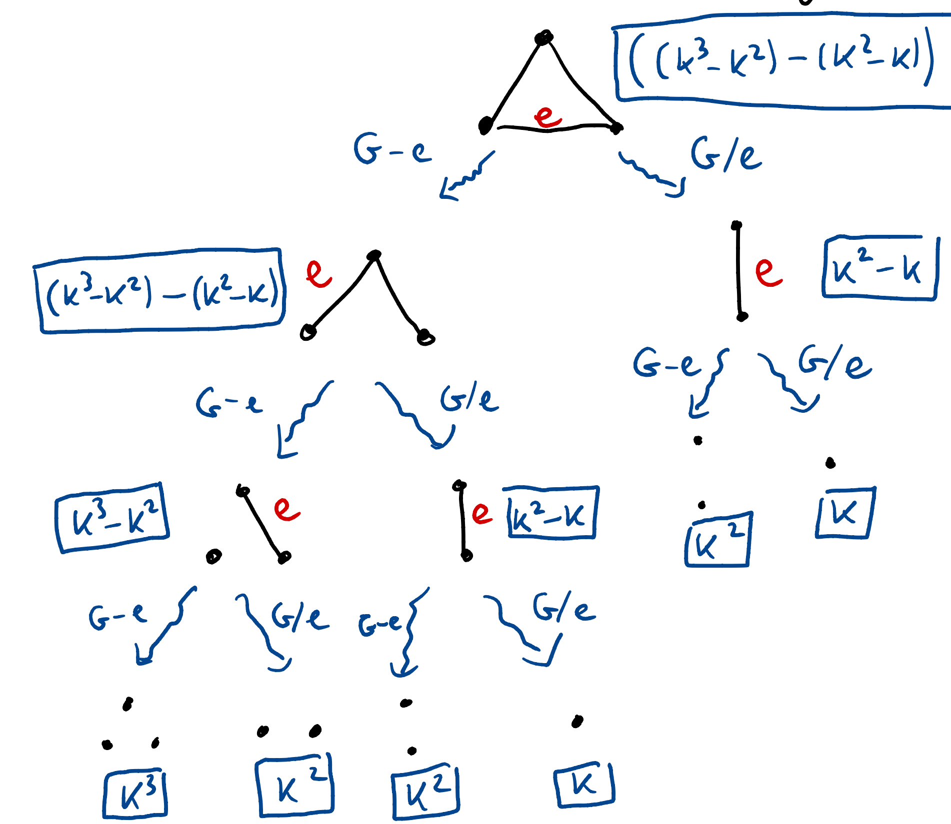


But... if we have a coloring of $G-e$ where u and v have the same color, then we can "squeeze" u and v together to obtain a proper coloring of G/e. So by subtracting $\chi(G/e,k)$ from $\chi(G-e,k)$ we cancel out "bad" colorings of G and get $\chi(G,k)$. ∎

Cor $\chi(G,k)$ is a polynomial in k, of degree $= \#\text{vertices}(G)$.

Pf: By deletion/contraction can repeatedly remove edges until we reduce to case of G = empty graph, which has $\chi(G,k) = k^{\#\text{vert's}(G)}$. ▨

This inductive way of computing $\chi(G,k)$ can be represented by a "tree" of graphs:



here we write $\boxed{\chi(G,k)}$ next to each graph G

Check: $((k^3-k^2)-(k^2-k)) - (k^2-k) = k^3-3k^2+2k$
$= k(k-1)(k-2)$ ✓ //