# Math 4707: acyclic orientations

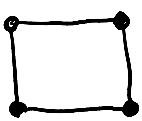
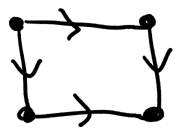
4/26

Not in LPV

Reminder: • The final exam has been posted. It is due Wednesday, May 5th.

We have officially covered all the material in the course that will be assessed on assignments. The last couple classes will be "bonus material!"

Today we will talk about acyclic orientations of graphs. Let $G$ be an (undirected) graph. An orientation $\mathcal{O}$ of $G$ is a choice for each edge $e=\{u,v\}$ of $G$ of one of the two orientations $(u,v)$ or $(v,u)$:

e.g. $G =$  $\rightsquigarrow$ an orientation $\mathcal{O}$ of $G$ = 

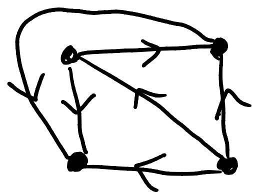
We can think of an orientation $\mathcal{O}$ as a directed graph whose underlying undirected graph is $G$.

We'll be interested in counting families of orientations of graphs. Counting all orientations is easy:

**Prop.** The # of orientations of $G$ is $2^{\#\text{edges}(G)}$.

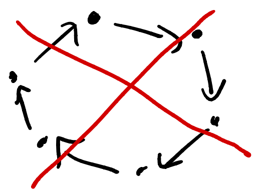
**Pf:** An orientation is defined by the choice of 1 of 2 things ($(u,v)$ or $(v,u)$) for each edge $e = \{u, v\}$. ☒

Recall that many classes ago we discussed tournaments, which are the same thing as orientations of the complete graph $K_n$:



← tournament on 4 vertices

Remember that a digraph is acyclic if it does not contain a directed cycle:



When we discussed tournaments, we explained why acyclic tournaments are the same as transative tournaments, and that there are $n!$ transitive tournaments on $n$ vertices (corresponding to orderings of vertices).

In other words, there are $n!$ acyclic orientations of the complete graph $K_n$.

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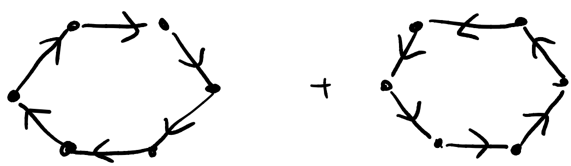
We will now focus on counting acyclic orientations (a.o.'s) of a fixed graph $G$.

e.g. $G = K_n \Rightarrow$ there are $n!$ a.o.'s of $G$, as we just saw

e.g. $G$ = a tree on $n$ vertices $\Rightarrow$ there are $2^{n-1}$ a.o.'s of $G$, since all orientations are acyclic

e.g. $G = C_n$, cycle graph on $n$ vertices $\Rightarrow$ there are $2^n - 2$ a.o.'s of $G$, since all orientations are acyclic except...



However, for other graphs it might be hard to count a.o.'s for other graphs $G$ by hand. Instead, we will give a recurrence formula based on the operations of deletion $G-e$ and contraction $G/e$ we defined last class.

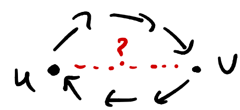
Set $ao(G) :=$ # a.o.'s of $G$.

**Thm** For any edge $e$ of $G$,
$$ao(G) = ao(G-e) + ao(G/e).$$

**Pf:** An acyclic orientation $\mathcal{O}'$ of $G-e$ is almost the same as an acyclic orientation $\mathcal{O}$ of $G$: we just have to choose how to orient $e = \{u,v\}$.

**Claim:** at least one of $(u,v)$ or $(v,u)$ gives an a.o. from $\mathcal{O}'$.

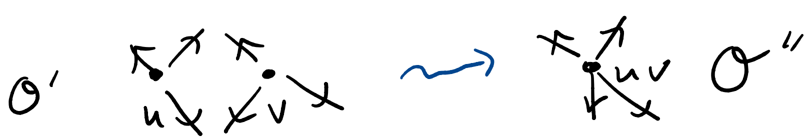
Otherwise... have directed paths $u \to v$ and $v \to u$:




That would contradict that $\mathcal{O}$ is acyclic.

It could be that both choices of $(u,v)$ and $(v,u)$ are okay. This is when $\mathcal{O}'$ has no paths $u \to v$ or $v \to u$.

In such a case we can "squeeze" $u$ and $v$ together in $\mathcal{O}'$ to produce an a.o. $\mathcal{O}''$ of $G/e$:




This means that

$$ao(G-e) + ao(G/e) = ao(G),$$

count all a.o.'s of $G-e$

count the a.o.'s of $G-e$ w/ 2 extensions to $G$ another time

as claimed. ∎

The recurrence $ao(G) = ao(G-e) + ao(G/e)$ looks a lot like the recurrence $\chi(G,k) = \chi(G-e,k) + \chi(G/e,k)$ we proved for the chromatic polynomial last class.

In fact...

**Thm** For $G$ graph on $n$ vertices,

$$ao(G) = (-1)^n \cdot \chi(G,-1).$$

plug $k=-1$ into chrom. poly.

**pf:** They satisfy same recurrence:

$$ao(G) = ao(G-e) + ao(G/e)$$

$$(-1)^n \chi(G,-1) = (-1)^n \left( \chi(G-e,-1) - \chi(G/e,-1) \right)$$

$G/e$ has $n-1$ vertices

$$= (-1)^n \chi(G-e,-1) + (-1)^{n-1} \chi(G/e,-1).$$

And both $=1$ in base case of $G =$ graph w/ no edges. ∎

**e.g.** For $G = K_n$ the complete graph, we saw last class

$$\chi(K_n,k) = k \cdot (k-1) \cdot (k-2) \cdots (k-(n-1)),$$

So $(-1)^n \chi(K_n,-1) = (-1)^n \cdot (-1) \cdot (-2) \cdot (-3) \cdots (-n) = n!$,

and we explained above why $ao(K_n) = n!$ ✓

**e.g.** For $G = P_n$ path graph on $n$ vertices, we saw last class

$$\chi(P_n,k) = k(k-1)^{n-1},$$

So $(-1)^n \chi(P_n,-1) = (-1)^n \cdot (-1) \cdot (-2)^{n-1} = 2^{n-1}$,

and we explained above why $a_o(P_n) = 2^{n-1}$. ✓

Surprising that a.o.'s = "colorings w/ $-1$ colors"! //