Math 4707: More P.I.E. and binomial coeff's

2/1

Ch.'s 2+3 of LPV

Reminder: • HW#1 due Wed. 2/3 - upload to Canvas

Last class we learned about the principle of inclusion-exclusion (P.I.E.): for $A_1, A_2, \ldots, A_m \subseteq U$,

$$\#(U - A_1 \cup \cdots \cup A_m) = \#U - \#A_1 - \#A_2 - \cdots - \#A_m$$
$$+ \#A_1 \cap A_2 + \#A_1 \cap A_3 + \cdots - \cdots + (-1)^m \# A_1 \cap A_2 \cap \cdots \cap A_m.$$

Before we prove the P.I.E., let's go over one more nice application of it.

Set partitions A (set) partition of $[n] = \{1, 2, \ldots, n\}$ is a set $\{B_1, B_2, \ldots, B_m\}$ of subsets $B_1, B_2, \ldots, B_m \subseteq [n]$ s.t.

- $B_i \neq \emptyset \ \forall i$ (nonempty)
- $B_i \cap B_j = \emptyset \ \forall i \neq j$ (pairwise disjoint)
- $B_1 \cup B_2 \cup \cdots \cup B_m = [n]$ (cover all of $[n]$)

The B_i are called the blocks (or parts) of the partition.

e.g., $\{\{1,3,4\}, \{2,6\}, \{5\}\} = 134\text{-}26\text{-}5$ is a partition of $[6]$ into 3 blocks.

Def. $S(n,k) :=$ # partitions of $[n]$ into k blocks

e.g. $S(3,2) = 3$ since partitions are 12-3, 13-2, 1-23.

$S(n,k)$ called Stirling #'s of the 2^{nd} kind.

For $S(n,k)$, the order of the blocks does not matter.

$\hat{S}(n,k) :=$ # ordered partitions of $[n]$ into k blocks

e.g. $\hat{S}(3,2) = 6$ since ordered par.'s: 12-3, 13-2, 1-23, 3-12, 2-13, 23-1

Now let's do a worksheet in breakout groups where we find formula for $\hat{S}(n,k)$ + $S(n,k)$ using P.I.E.

Note: Skip #3, 4, + 5 on worksheet.

Worksheet # ways to put n distinct balls in k distinct bins s.t. every bin has ≥ 1 ball

$\Rightarrow \hat{S}(n,k) =$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \quad \text{(P.I.E.)}$$

$$S(n,k) = \frac{1}{k!} \hat{S}(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

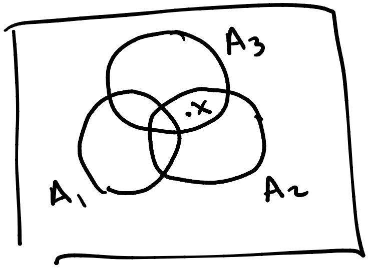
Now let's prove the P.I.E.

Thm For $A_1, A_2, \ldots, A_m \subseteq U$,

$$\#\left(U - \bigcup_{i=1}^{m} A_i\right) = \#U - \sum_{\emptyset \neq \{i_1, i_2, \ldots, i_k\} \subseteq [m]} (-1)^k \# A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}.$$

Pf: Take any $x \in U$. How much does x contribute to RHS of formula?

e.g.



$+1 \quad x \in U$

$-1 \quad x \in A_2, \quad -1 \quad x \in A_3$

$+1 \quad x \in A_2 \cap A_3$

0 total contribution

If x belongs to exactly ℓ of the $A_1, A_2, \ldots, A_m$ then x contributes $\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}$ to the R.H.S.

e.g. If $\ell = 2$, contr. $= +\binom{2}{0} - \binom{2}{1} + \binom{2}{2} = +1 - 2 + 1 = 0$ ✓

Claim $$\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} = \begin{cases} 0 & \text{if } \ell > 0, \\ 1 & \text{if } \ell = 0. \end{cases}$$

Claim proves thm., since this is how much x contributes to L.H.S. (only want to count the x w/ $\ell = 0$, i.e., belonging to none of A_i). ∎

But how to prove Claim?

Let's finally explain the name binomial coefficients.

Thm (Binomial Theorem) For $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$

e.g., $(x+y)^3 = y^3 + 3y^2x + 3yx^2 + x^3$

$$= \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0.$$

Pf: Do the expansion

$$\underbrace{(x+y)(x+y) \cdots (x+y)}_{n \text{ copies of } (x+y)}$$

To get a term of $x^k y^{n-k}$, need to pick x from k of the $(x+y)$'s and y from the remaining $n-k$ $(x+y)$'s. The # of ways to do this is of course $\binom{n}{k}$. $\square$

Rmk: Multinomial coeff's $\binom{n}{k_1, k_2, \ldots, k_m}$

Similarly come from $(x_1 + x_2 + \cdots + x_m)^n$.

Now let's get back to the claim:

Cor. $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0 & \text{if } n>0, \\ 1 & \text{if } n=0. \end{cases}$

pf: $\displaystyle(-1+1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$ by binomial thm.

$$\begin{array}{l} \| \\ 0^n = \begin{cases} 0 & \text{if } n>0 \\ 1 & \text{if } n=0, \text{ since } 0^0 = 1 \end{cases} \end{array}$$

???

∎

The binomial thm suggests we study

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n}$$

in a row like that.

e.g. $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4} = 1, 4, 6, 4, 1$

Actually there's a way to fit all of the $\binom{n}{k}$ into a very pretty/useful array called Pascal's triangle:

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\vdots$$
$$\parallel$$
$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$\ddots$$

There are many patterns in Pascal's triangle. Do you notice any patterns?

Symmetry!

Thm $\binom{n}{k} = \binom{n}{n-k}$

Pf: Follows from $\binom{n}{k} = \frac{n!}{k!(n-k)!}$,

but can you think of a bijective proof? $\boxtimes$

We saw that alternating sum of n^{th} row of Pascal's Δ is

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n} = \begin{cases} 0 & n>0 \\ 1 & n=0 \end{cases}$$

What about just normal sum of n^{th} row?

Thm $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

Pf: Do you see a proof? ... $\boxtimes$

Other patterns abound, like

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

See the book for this + more ...

Most important pattern in Pascal's Δ is Pascal's identity:

<u>Thm</u> $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

<u>e.g.</u>

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

$$\begin{array}{ccc} \binom{n-1}{k-1} & & \binom{n-1}{k} \\ & \searrow \quad \swarrow & \\ & \binom{n}{k} & \end{array}$$

<u>Note</u>: Let's you easily fill in Pascal's Δ!

<u>Pf</u>: Let's define a bijection

$$f: \left\{\begin{matrix} k\text{-subsets} \\ \text{of } [n] \end{matrix}\right\} \to \left\{\begin{matrix} k\text{-subsets} \\ \text{of } [n-1] \end{matrix}\right\} \cup \left\{\begin{matrix} k-1\text{-subsets} \\ \text{of } [n-1] \end{matrix}\right\}$$

by $$f(A) = \begin{cases} A & \text{if } n \notin A \quad (\text{a } k\text{-subset of } [n-1]) \\ A \setminus \{n\} & \text{if } n \in A \quad (\text{a } k-1\text{-subset of } [n-1]) \end{cases}$$

<u>This</u> exactly corresponds to Pascal's identity. $\boxtimes$

<u>Rmk</u>: We have a similar identity

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$

for Stirling #'s of 2^{nd} kind, w/ a very similar bijective proof.

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