Math 4707: Intro to combinatorics and graph theory
Spring 2021, Instructor: Sam Hopkins
Midterm exam 1- Due Wednesday Feb. 24th

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are not allowed to to interact with anyone (including online forums) except for me, the instructor. As always, in order to earn points you need to carefully *explain your answer.*

1. (20 points total)
   (a) (5 points) How many rearrangements (i.e., anagrams) are there of the letters in the word "COMMITTEE"?
   (b) (15 points) What's the probability a random such rearrangement has no identical letters consecutive (no "MM", "TT", nor "EE")?

2. (20 points) Define the sequence of numbers $P_0, P_1, P_2, \ldots$ via initial conditions $P_0 = 0$, $P_1 = 1$, and recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Find $\lim_{n\to\infty} \frac{P_{n+1}}{P_n}$.

3. (20 points) Suppose that X is a subset of $\{1, 2, ..., 2n\}$ of size $n + 1$. Show that there must be two numbers a and b in X such that a and b are relatively prime (i.e., $\gcd(a, b) = 1$).
   Hint: use the Pigeonhole Principle!

4. (20 points) Exercise 1.8.29 on p. 24 of our text: In how many ways can one color n distinct objects (labeled $1, 2, \ldots, n$) with 3 colors, if each color must be used at least once? (Your answer should be expressed as a function of n.)

5. (20 points) Exercise 1.8.32 on p. 24 of our text: Find all triples (a, b, c) of positive integers with $a \geq b \geq c \geq 1$ such that

$$\binom{a}{b}\binom{b}{c} = 2\binom{a}{c}.$$