UMN MATH 8668 FALL 2019 BONUS PROBLEMS

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(1) (a) [2] Let $\mathrm{ODD}(n)$ denote the permutations in $\mathfrak{S}_n$ with no cycles of even length. Show (for instance, using generating function manipulations) that

$$\sum_{\sigma \in \mathrm{ODD}(n)} 2^{\#\mathrm{cycles}(\sigma)-1} = n!$$

for all $n \geq 1$.

(b) [2+] Prove the equality in part (a) bijectively.

(2) (a) [2] Recall the *q-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{w \text{ rearrangement of} \\ k \text{ 1's, } n-k \text{ 0's}}} q^{\mathrm{inv}(w)}.$$

Assume that $n = 2N$ **is an even number**. Show, **by using a sign-reversing involution**, that the evaluation of the q-binomial coefficient at $q := -1$ is equal to the number of rearrangements $w = (w_1, w_2, \ldots, w_n)$ of k 1's and $n-k$ 0's for which $w_i = w_{n+1-i}$ for all $1 \leq i \leq n$. (In other words, setting

$$W := \{\text{rearrangements } w \text{ of } k \text{ 1's and } n-k \text{ 0's }\},$$

find an involution $\tau \colon W \to W$ such that

- $\mathrm{inv}(w)$ and $\mathrm{inv}(\tau(w))$ have opposite parity whenever $w \neq \tau(w)$;
- $\mathrm{inv}(w)$ is even if $w = \tau(w)$;
- $\#\{w \in W \colon w = \tau(w)\} = \#\{w \in W \colon w_i = w_{n+1-i}\, \forall 1 \leq i \leq n\}$.)

(b) [2] Adapt your argument from (a) to give a combinatorial interpretation of the evaluation at $q := -1$ of the *q-multinomial coefficient*

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q = \sum_{\substack{w \text{ rearrangement of} \\ k_1 \text{ 1's, } k_2 \text{ 2's, } \ldots, \, k_\ell \ \ell\text{'s}}} q^{\mathrm{inv}(w)},$$

(still under the assumption that $n = 2N$ is even).

(3) (a) [2-] Let P be a finite poset. Let $\mathcal{A}$ denote the set of antichains of P (recall that an *antichain* of P is a subset $A \subseteq P$ of pairwise incomparable elements). Define a partial order $\preceq$ on $\mathcal{A}$ by $X \preceq Y$ iff for all $x \in X$ there exists $y \in Y$ such that $x \leq_P y$. Show that $(\mathcal{A}, \preceq)$ is isomorphic to $J(P)$, the distributive lattice of order ideals of P.

(b) [2] Let $(A_1, A_2, \ldots, A_k) \in \mathcal{A}^k$ be any k-tuple of antichains. Show that we can always find a k-element multichain $B_1 \preceq B_2 \preceq \cdots \preceq B_k$ in $(\mathcal{A}, \preceq)$ so that $\#\{i = 1, \ldots, k \colon p \in A_i\} = \#\{i = 1, \ldots, k \colon p \in B_i\}$ for all $p \in P$ (i.e., the multiset sum of the A_i equals the multiset sum of the B_i). Show that this multichain $B_1 \preceq \cdots \preceq B_k$ is unique.