Fall 2019, UMN Math 8668:

Combinatorial Theory (1^{st} sem. intro grad comb.)

Instructor: Sam Hopkins, shopkins@umn.edu

website: umn.edu/~shopkins/classes/8668.html

~~[scribble]~~ 9/4/19 Class info:
- meets MWF 2:30-3:20, Ford Hall B60
- office hrs: MW 10-11am, Vincent Hall 204 (or by appointment if you email me...)
- Text: R. Stanley's "Enumerative Combinatorics, Vol 1" (link to pdf on website) HW problems mainly from here

  I'll also heavily use notes of F. Ardila (they are great! :)) (also linked to on website)
- Grading: There will be 3 HW's (roughly: Oct, Nov, Dec) Beyond that, show up to class and be engaged (e.g. ask questions) to get a good grade.
- This is the 1^{st} half of a year-long sequence. 8669 will be taught by Chris Fraser in the Spring

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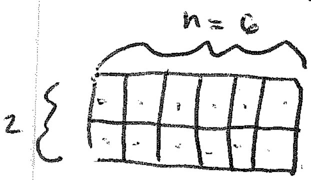
<u>What is this class about?</u>

We want to <u>count</u> combinatorial objects (subsets, multisets, partitions, graphs, etc.) and more generally understand their <u>structure</u> (e.g. partial order structure).

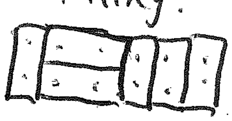
§1.1

What is a "good" answer to a counting problem?
It depends! (For instance, on what we want to do w/ answer...)
Rather than try to formalize a notion of "good answer" let's explore what answers can look like in an example.

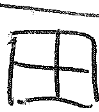
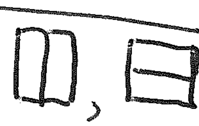
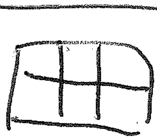
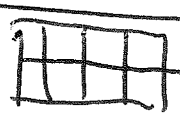
Ex: Let a_n = # tilings of $2 \times n$ rectangle by dominoes



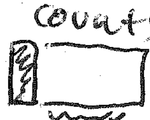
tiling:

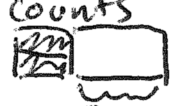


either  or 

n	a_n	rectangle	tilings
0	1		
1	1		
2	2		
3	3		
4	5		 a_3 a_2

① Recurrence! $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, w/ $a_0 = a_1 = 1$

a_{n-1} counts  $n-1$

a_{n-2} counts  $n-2$

This is the same recurrence as the Fibonacci numbers

$$F_n = F_{n-1} + F_{n+2} \quad \text{for } n \geq 2$$

Usually F_i are indexed w/ $F_0 = 0$, $F_1 = 1$

n	a_n	F_n
0	1	0
1	1	1
2	2	1
3	3	2
4	5	3

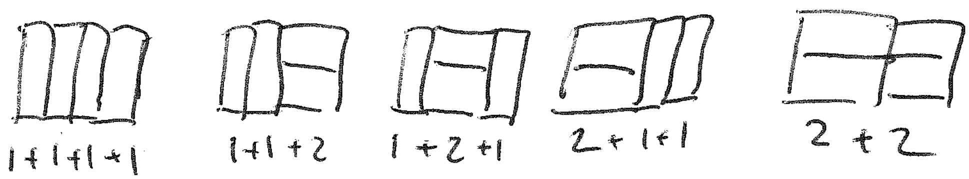
← So $a_n = F_{n+1}$, i.e., we're just considering Fibonacci numbers w/ slightly different indexing...

This recurrence lets us compute a_n, but it might take a while + we don't get a sense of growth rate of a_n.

② Explicit formula as a summation

Observe $a_n = \#\{\text{sequences of 1's and 2's summing to } n\}$

e.g. $n = 4$



$$\Rightarrow a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \#\{\text{sequences of } k \text{ 2's and } n-2k \text{ 1's}\}$$

e.g. $\quad 1 + 2 + 1 + 1 + 2 + 1 + 2 = n = 10$

$k = 3$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{(n-2k)+k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$$

e.g. $a_4 = \binom{4}{0} + \binom{4-1}{1} + \binom{4-2}{2} = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 1+3+1 = 5$

This formula is "explicit", but still not so fast to compute + still does not give sense of growth.

Also, $\exists$ of this formula doesn't mean there isn't a better one

e.g. # subsets of $\{1,2,\ldots,n\} = \sum_{k=0}^{n} \binom{n}{k}$ ("explicit" formula)

$= 2^n$ (better formula!)

3) Explicit formula w/ exponentiation

The recurrence relation $a_n = a_{n-1} + a_{n-2}$ (w/ initial conditions $a_0 = a_1 = 1$) implies that

$$a_n = \frac{1}{\sqrt{5}}\left(\Big(\underbrace{\frac{1+\sqrt{5}}{2}}_{\varphi}\Big)^{n+1} - \Big(\underbrace{\frac{1-\sqrt{5}}{2}}_{\psi}\Big)^{n+1}\right).$$

(You may have seen this e.g. in a linear algebra course.)

We'll explain why this formula holds very soon.

This formula is very explicit, and it does let us see growth rate of a_n, but it has its own drawbacks: e.g. why is it even an integer?

4) Asymptotic Formula

Can compute $\begin{cases} \varphi \approx 1.618\ldots & (\text{"golden ratio"}) \\ \psi \approx -0.618\ldots \end{cases} \Rightarrow |\varphi| > |\psi|$,

which means $a_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$, telling us a lot about its growth rate; e.g., # digits of a_n is

$$\log_{10}(a_n) \approx (n+1)\underbrace{\log_{10}(\varphi)}_{\approx 0.2} + \underbrace{\log_{10}\left(\tfrac{1}{\sqrt{5}}\right)}_{\text{small}}$$

(5) (Ordinary) generating function for a_n

$$A(x) \overset{\text{DEF'N}}{:=} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\left(= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots\right)$$

$$= \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]],$$

the ring of formal power series in x, with coefficients in $\mathbb{C}$.

(we'll give a formal definition of their algebraic structure in a short while...)

Not so clear at first why you'd ever consider $A(x)$, but we'll see that generating functions are extremely powerful, e.g., we can derive everything we saw about (a_n) from $A(x)$.

9/6/19

Claim: $A(x) = \dfrac{1}{1-x-x^2}$

Pf: Recall Recurrence $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ (and $a_0 = a_1 = 1$)

Multiply by x^n and sum over all $n \geq 2$ to get:

$$\sum_{n\geq 2} a_n x^n = \sum_{n\geq 2} a_{n-1} x^n + \sum_{n\geq 2} a_{n-2} x^n$$

$$A(x) - a_0 x^0 - a_1 x^1 = x\left(\sum_{m\geq 1} a_m x^m\right) + x^2\left(\sum_{m\geq 0} a_m x^m\right)$$

$$A(x) - 1 - x = x\left(A(x) - a_0 x^0\right) + x^2\left(A(x)\right)$$

$$(1-x-x^2)\,A(x) = x + 1 - x = 1$$

$$\Rightarrow \quad A(x) = \frac{1}{1-x-x^2}$$

What good is knowing $A(x) = \frac{1}{1-(x+x^2)}$? Plenty!

Let's extract coefficients of $A(x)$ in various ways...

(a) $A(x) = \frac{1}{1-(x+x^2)} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots$

i.e., $\sum_{n\geq 0} a_n x^n = \sum_{d\geq 0} (x+x^2)^d = \sum_{d\geq 0} \sum_{k=0}^{d} \binom{d}{k} \underbrace{(x^2)^k x^{d-k}}_{x^{d+k}}$

$$= \sum_{n\geq 0} x^n \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \right)$$ $n = d+k$, $d = n-k$

$\Rightarrow a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$, our first explicit formula from before

(b) $A(x) = \frac{1}{1-x-x^2} = \frac{\frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)}{1-\frac{1+\sqrt5}{2}x} + \frac{\frac{-1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)}{1-\frac{1-\sqrt5}{2}x}$

How to see this?

Recall partial fractions: $\frac{1}{ax^2+bx+c} = \frac{1}{a(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2}$

$$= \frac{-A/r_1}{1-\frac{x}{r_1}} + \frac{-B/r_2}{1-\frac{x}{r_2}}$$

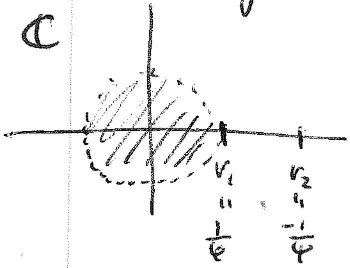
Here $r_1 = \left(\frac{1+\sqrt5}{2}\right)^{-1}$, $r_2 = \left(\frac{1-\sqrt5}{2}\right)^{-1}$

$\Rightarrow = \frac{1}{\sqrt5} \sum_{n\geq 0} \left(\frac{1+\sqrt5}{2}\right)^{n+1} x^n - \frac{1}{\sqrt5} \sum_{n\geq 0} \left(\frac{1-\sqrt5}{2}\right)^{n+1} x^n$

$\Rightarrow a_n = \frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^{n+1} - \left(\frac{1-\sqrt5}{2}\right)^{n+1}\right)$, our second explicit formula from before...

Ⓒ The asymptotic $a_n \approx c \left(\overbrace{\frac{1+\sqrt{5}}{2}}^{r_1^{-1}}\right)^n$ for some constant c

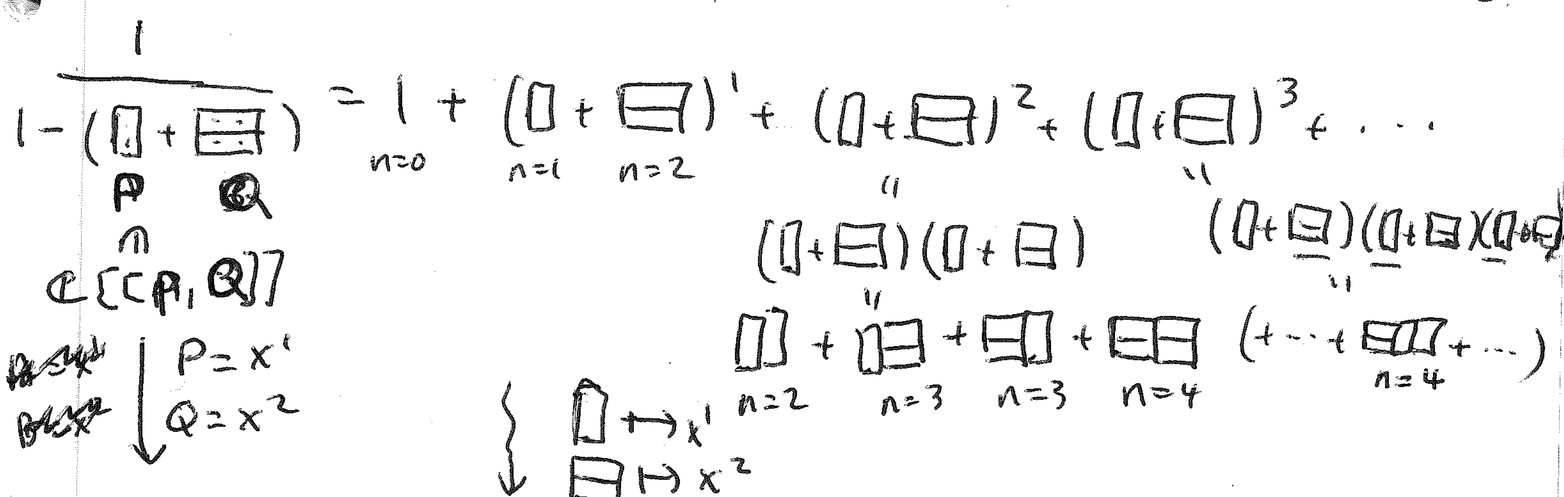
corresponds to the fact that r_1^{-1} is the reciprocal of the pole of $A(x) = \frac{1}{1-x-x^2} = \frac{1}{(x-r_1)(x-r_2)}$ <u>nearest</u> the origin of $\mathbb{C}$.



This is just the tip of the rich interplay between thinking of $A(x)$ as a <u>formal</u> power series, and as a <u>(analytic) function</u> of a <u>complex variable</u>.

(for more see H. Wilf's "generatingfunctionology", linked to on website of the class.)

The fast way to get $A(x) = \frac{1}{1-x-x^2}$ is via Polya's "<u>picture-writing</u>":



$\mathbb{C}[[x]] \ni A(x) = \frac{1}{1-(x+x^2)} \quad \left(= 1 + (x+x^2)^1 + (x+x^2)^2 + \cdots \right)$

The generating function can often be <u>refined</u> to keep track of additional <u>statistics</u> on our combinatorial objects.

Say we want to compute

$$a_{m,n} = \#\{\text{tilings of } 2\times n \text{ rectangle w/ } m \text{ vertical tiles}\}$$

from "picture-writing" we get

"weight" vertical tile by v, formal parameter

$$\left[\frac{1}{1-(\underset{P}{\square}+\underset{Q}{\boxminus})}\right]_{\substack{P=vx,\\ Q=x}} = \sum_{n,m\geq 0} a_{m,n}\, x^n v^m \in \mathbb{C}[[x,v]]$$

$$\left[\frac{1}{1-(P+Q)}\right]_{\substack{P=vx,\\ Q=x}} = \frac{1}{1-vx-x^2}$$

This g.f. is useful for e.g. computing (asymptotically) the <u>expected</u> <u>number</u> of vertical tiles in a random tiling

$$\sum_{n\geq 0}\left(\sum_{m\geq 0} a_{m,n}\cdot m\right)x^n = \left[\partial/\partial v \sum_{n,m\geq 0} a_{m,n}\, x^n v^m\right]_{v=1}$$

$$= \left[\partial/\partial v\ \frac{1}{1-vx-x^2}\right]_{v=1}$$

$$= \left[\frac{x}{(1-vx-x^2)^2}\right]_{v=1} = \frac{x}{(1-x-x^2)^2}$$

Via partial fractions $\left(\frac{x}{(1-x-x^2)} = \frac{A_1x+B_1}{(x-r_1)^2} + \frac{A_2x+B_2}{(x-r_2)^2} + \frac{C}{(x-r_1)} + \frac{D}{(x-r_2)}\right)$

Can use above formula to show $\sum_{m\geq 0} a_{m,n}\, m \approx \frac{n}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$

Recall $a_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \Rightarrow \sum_{m\geq 0} a_{m,n}\, m \approx \frac{n}{\sqrt{5}}\cdot a_n$

Thus, the expected # of vertical tiles is $\approx \frac{n}{\sqrt{5}}$,

i.e., out of the n tiles, about $\frac{1}{\sqrt{5}}$ of them will be vertical.

9/9/2019 ~~The ring of formal power series~~ $R[[x]]$

(where $R=\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{C}[v]$ or any commutative ring w/ 1)
or $\mathbb{F}_q$

DEF'N $R[[x]] := \{a_0 + a_1 x + a_2 x^2 + \cdots = \overset{A(x)}{\sum_{n=0}^{\infty} a_n x^n}$ w/ $(a_0, a_1, a_2, \ldots) \in R\}$

is a commutative ring w/ coefficientwise addition: for $B(x) = \sum_{n=0}^{\infty} b_n x^n$, $A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$,

and multiplication via convolution:

$$C(x) := A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

So its zero is $0 = 0 + 0x + 0x^2 + \cdots$

and its one is $1 = 1 + 0x + 0x^2 + 0x^3 + \cdots$

Prop. $A(x) = \sum_{n=0}^{\infty} a_n \in R[[x]]$ is a unit (i.e., $\exists B(x)$ w/ $1 = A(x)B(x)$)

$\Longleftrightarrow a_0$ is a unit of R (i.e., $\exists\, b_0 \in R$ w/ $1 = a_0 b_0$).

E.g. By this criterion, $(1-x-x^2) \in \mathbb{C}[[x]]$ is a unit, so $\exists A(x)$ w/ $A(x)(1-x-x^2) = 1$, i.e. $A(x) = \frac{1}{1-x-x^2}$ $(= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots)$

Proof:

$$\underset{1+0x+0x^2+\cdots}{1} = A(x)B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

$\Longleftrightarrow a_0 b_0 = 1$ (i.e., $b_0 = a_0^{-1}$ in R) (so a_0 needs to be a unit for $A(x)$ to possibly be one)

and then

$a_0 b_1 + a_1 b_0 = 0$ means $b_1 = \frac{-a_1 b_0}{a_0}$ ← allowed since $b_0 = a_0^{-1}$ already defined

$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$ means $b_2 = \frac{-(a_1 b_1 + a_2 b_0)}{a_0}$

... (can recursively compute all b_i in unique way) ∎

DEF'N A sequence $A_1(x), A_2(x), \dots$ in $R[[x]]$ <u>converges</u> (i.e., $\exists A(x)$ w/ $A(x) = \lim_{y\to\infty} A_y(x)$) if $\forall n \geq 0$, the <u>coefficient of x^n</u> in $A_j(x)$ stabilizes for $j \gg 0$;

denote this by $[x^n]A_j(x)$

i.e., $\forall n \geq 0$, $\exists N > 0$ and $a_n \in R$ s.t. $[x^n]A_j(x) = a_n$ $\forall j \geq N$.

e.g. $$A(x) = \frac{1}{1-(x+x^2)} = \underbrace{\underbrace{\underbrace{\underbrace{1}_{A_0(x)} + (x+x^2)}_{A_1(x)} + (x+x^2)^2}_{A_2(x)} + (x+x^2)^3}_{A_3(x)} + \dots$$

converges in $\mathbb{C}[[x]]$, e.g. $[x^3]A(x) = [x^3]A_3(x) = [x^3]A_4(x) = \dots = a_3 = 3$.

e.g. $e^{x+1} := 1 + \frac{(x+1)}{1!} + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \dots$ does not converge (i.e., $e^{x+1} \notin R[[x]]$) even if it makes sense analytically,

but $e^x := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ does converge.

Alternatively, $\{A_j(x)\}_{j=0,1,\dots}$ converges in $R[[x]]$,

if $\lim_{j\to\infty} \underset{\text{deg}}{\text{min}} (A_j(x) - A_{j-1}(x)) = \infty$

where <u>DEF'N</u> mindeg $\underset{\sum_{n\geq 0} a_n x^n}{\overset{\shortparallel}{A(x)}} :=$ smallest d w/ $a_d \neq 0$ ($= \infty$ if no such d)

e.g. $A(x) = \frac{1}{1-x-x^2}$, then $A_j(x) - A_{j-1}(x) = (x+x^2)^j$

and min deg $(x+x^2)^j = j \to \infty$ as $j \to \infty$.

$\underline{\underline{\text{Cor}}}$ $\sum_{j=0}^{\infty} B_j(x) = \underbrace{\underbrace{\underbrace{B_0(x)}_{A_0(x)} + B_1(x)}_{A_1(x)} + B_2(x)}_{A_2(x)} + \cdots$ converges in $R[[X]]$

$\iff$ mindeg $B_j(x) \to \infty$ as $j \to \infty$.

$\| \lim_{n\to\infty} A_n(x)$

$[$w/ $B_j = A_j - A_{j-1}]$

$\underline{\underline{\text{Cor}}}$ Infinite products of the form

$\underbrace{\prod_{j=1}^{\infty} (1 + B_j(x))}$ w/ mindeg $B_j \geq 1$ $\forall j$

converges in $R[[x]]$ $\iff$ $\lim_{j\to\infty}$ mindeg $B_j = \infty$

$\left(= \lim_{n\to\infty} A_n(x) \text{ where } A_0 = 1,\ A_1 = (1 + B_1(x)),\ A_2 = (1+B_1(x))(1+B_2(x)), \ldots \right)$

$\underline{\text{E.g.}}$ $\prod_{n=1}^{\infty}(1 + \frac{1}{2^n} x)$ $\underline{\text{does not converge}}$ in $R[[x]]$
$\quad \hookrightarrow A(x)$

(even if it does make sense analytically to think of $A(x)$ as a function of $x \in \mathbb{C}$ for at least $|x|$ small enough)

$\underline{\text{E.g.}}$ $\prod_{n=1}^{\infty}(1 + x^n)$ converges in $R[[x]]$

$\|$

$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$

$\|$

$\sum_{n\geq 0}^{\infty} a_n x^n$ $\qquad$ $\underline{\text{Q}}$: what are these coefficients a_n?

$\underline{\text{A}}$: We'll see next time...

1/16/19

Partition generating functions

DEF'N A (number) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of n is a • weakly decreasing eventually 0 sequence of nonnegative integers (i.e., $\lambda_i \in \mathbb{N}$ $\forall i = 1, 2, \dots$) $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq 0$ with $\lambda_1 + \lambda_2 + \dots = n$.

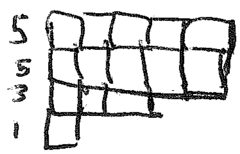
We write $\lambda \vdash n$ (λ \vdash n) and $|\lambda| = n$. ← "size" of λ

e.g. $\lambda = (5,5,3,1,0,0,\dots) = (5,5,3,1,0) = (5,5,3,1) \vdash 14 = 5+5+3+1$.

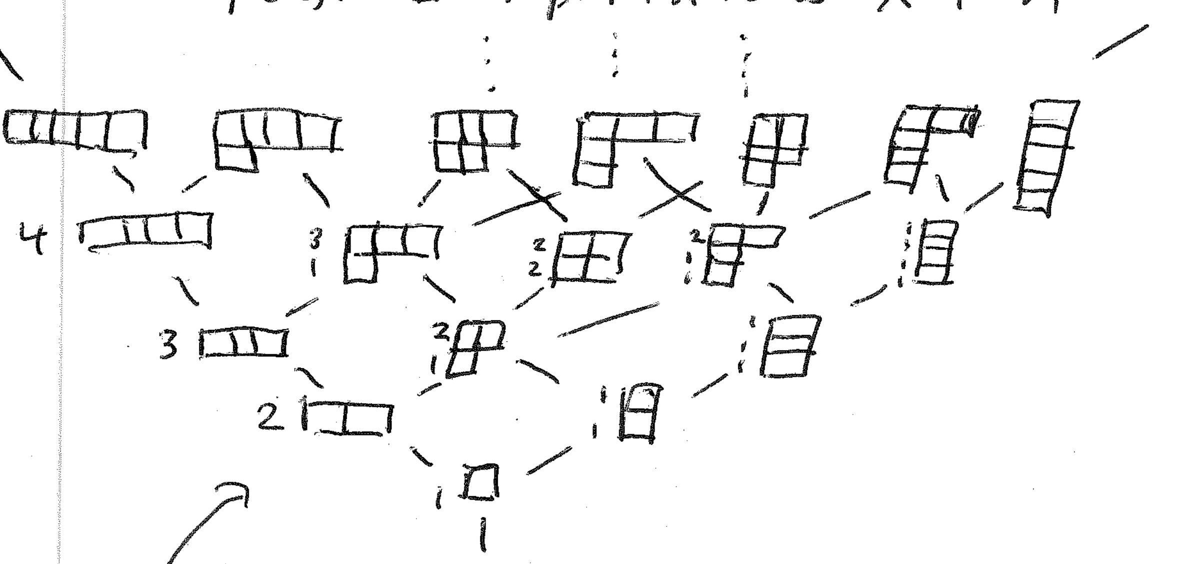
Its length $\ell(\lambda) := \{i : \lambda_i > 0\} = \#$ of nonzero parts λ_i

Its Young diagram is a left + top justified array of boxes with λ_i boxes in the i^{th} row from the top:

e.g. $\lambda = (5,5,3,1) \longleftrightarrow$



– Let $p(n) := \#$ of partitions $\lambda \vdash n$

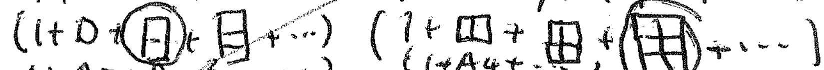


$\emptyset$ (empty partition, unique partition of 0)

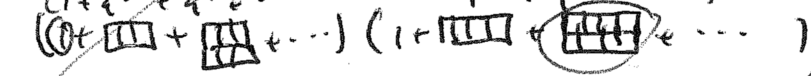
$p(n)$	n
⋮	⋮
7	5
5	4
3	3
2	2
1	1
1	0

$\mathbb{Y}$ = Young's lattice, the poset of all partitions ordered by containment of Young diagrams

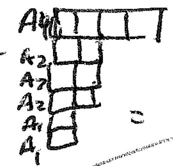
$$\sum_{\substack{n\geq 0 \\ \text{in } \mathbb{C}[[q]]}} p(n)\, q^n = \sum_{\substack{\text{all partitions} \\ \lambda}} q^{|\lambda|} = \begin{bmatrix} (1 + A_1 + A_1^2 + A_1^3 + \cdots) & (1 + A_2 + A_2^2 + A_2^3 + \cdots) \\ q^0 + q^1 + q^2 + q^3 + \cdots & (q^0 + q^2 + q^4 + q^6 + \cdots) \end{bmatrix}$$



$$(1 + A_3 + A_3^2 + \cdots) \quad (1 + A_4 + \cdots)$$
$$(1 + q^3 + q^6 + \cdots) \quad (1 + q^4 + q^8 + \cdots)$$



$$= \left[+ \cdots + \right.$$



$$\left. = A_1^2 A_2^4 A_3^0 A_4^1 \quad + \cdots \right]$$

$$A_i \mapsto q^i$$

$$= \underbrace{(1 + q + q^2 + q^3 + \cdots)}_{B_1} \underbrace{(1 + q^2 + (q^2)^2 + \cdots)}_{B_2} \underbrace{(1 + q^3 + (q^3)^2 + \cdots)}_{B_3} \cdots$$

$$= \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdots \quad \text{a \underline{convergent} \underline{product}!}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} . \longleftarrow \text{g.f. for all partitions as a \underline{product formula}!}$$

[Cultural asides:

Thm (Euler's pentagonal # thm)

$$\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x^{k(3k+1)/2} + x^{k(3k-1)/2} \right)$$

"pentagonal #'s"

Pf: Look up. Either Euler's original algebraic proof via g.f.'s, or very nice bijective pf due to Franklin.

Together with above g.f. for partitions get $\sum p(n) q^n = \left(\prod (1-q^n)\right)^{-1}$

$\Rightarrow$ Recurrence for $p(n) = \sum_{k = -\infty, k \neq 0}^{\infty} (-1)^k p(n - k(3k-1)/2)$ $\leftarrow$ [$p(m) = 0$ if $m < 0$]

$$= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + \ldots$$

Thm (Hardy + Ramanujan) $p(n) \approx \dfrac{1}{4n\sqrt{3}}\, e^{\pi \sqrt{\frac{2n}{3}}}$

Pf: "Circle method", careful analysis of singularities of $\prod \frac{1}{(1-q^n)}$, residues, etc.]

G.f.'s for restricted partitions:

Let $q(n) :=$ # of partitions of n into distinct parts.

n	$q(n)$	
0	1	$\emptyset$
1	1	
2	1	
3	2	
4	2	
5	3	

$$Q(q) := \sum_{n \geq 0} q(n)\, q^n = (1+q)(1+q^2)(1+q^3)(1+q^4)\cdots = \underbrace{\prod_{j \geq 1} (1+q^j)}_{\text{converges in } \mathbb{C}[[q]]}$$

Let $P_{odd}(n) :=$ # partitions of n into odd parts

n	$P_{odd}(n)$	
0	1	$\emptyset$
1	1	
2	1	
3	2	
4	2	
5	3	

Looks like $q(n) = P_{odd}(n)$, but how to show it?

G.f.'s! $P_{odd}(q) = \sum_{n \geq 0} P_{odd}(n)\, q^n = (1+q+q^2+\cdots)(1+q^3+q^6+\cdots)(1+q^5+q^{10}+\cdots)\cdots$

converges in $\mathbb{C}[[q]]$ $= \dfrac{1}{(1-q)(1-q^3)(1-q^5)\cdots} = \dfrac{1}{\prod_{j \geq 0} (1-q^{2j+1})}$ $\left(\overset{?}{=} \prod_{j \geq 1} (1+q^j) \right)$

How to show $Q(q) = P_{odd}(q)$?

Well, $Q(q) = (1+q)(1+q^2)(1+q^3)\cdots$

$$= \frac{(1-q^2)}{(1-q)} \cdot \frac{1-(q^2)^2}{1-q^2} \cdot \frac{1-(q^3)^2}{1-q^3} \cdots$$

$$= \frac{(1-\cancel{q^2})(1-\cancel{q^4})(1-\cancel{q^6})(1-q^8)\cdots}{(1-q)(1-\cancel{q^2})(1-q^3)(\cancel{1-q^4})\cdots}$$

$$= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} = P_{odd}(q) \;!$$

Was that manipulation allowed? Yes! Let's think of it slightly differently...

Let $R(q) := (1-q)(1-q^3)(1-q^5)\cdots = \frac{1}{P_{odd}(q)} \in \mathbb{C}[[q]]$,

Want to show $1 \stackrel{?}{=} Q(q)\,R(q)$ in $\mathbb{C}[[q]]$.

$$(1+0\cdot q+0\cdot q^2\cdots) = \Big((1+q)\underbrace{(1+q^2)(1+q^3)\cdots}_{\text{converges}}\Big)\Big((1-q)\underbrace{(1-q^3)(1-q^5)\cdots}_{\text{converges}}\Big)$$

$$= \underbrace{(1+q)(1-q)}_{(1-q^2)}\Big((1+q^2)(1+q^3)\cdots\Big)\Big((1-q^3)(1-q^5)\cdots\Big)$$

$$= (1-q^4)\cdot\Big(\underbrace{(1+q^3)}(1+q^4)\cdots\Big)\Big(\underbrace{(1-q^3)}(1-q^5)\cdots\Big)$$

$$= \underbrace{(1-q^4)(1-q^6)}\cdot\Big(\underbrace{(1+q^4)}(1+q^5)\cdots\Big)\Big((1-q^5)(1-q^7)\cdots\Big)$$

$$= \underbrace{(1-q^8)(1-q^6)\cdot\Big((1+q^5)(1+q^6)\cdots\Big)\Big((1-q^5)(1-q^7)\cdots\Big)}_{=1+0\cdot q+0\cdot q^2+0\cdot q^3+0\cdot q^4\cdots}$$

$$= \cdots \text{ etc.}$$

hinted at by this manipulation

$\exists$ Bijective proofs of $q(n) = P_{odd}(n)$ as well

(See Stanley Prop. 1.8.5 for two such proofs!) Basic idea is binary expansion;

$\in q(n)$

e.g. $\lambda = (9^5, 5^{12}, 3^2, 1^3) = (9^{2^0+2^2}, 5^{2^2+2^3}, 3^{2^1}, 1^{2^0+2^1}) \leftrightarrow \mu = (9\cdot 2^0, 9\cdot 2^2, 5\cdot 2^2, 5\cdot 2^3, 5\cdot 2^3, 3\cdot 2^1, 1\cdot 2^0, 1\cdot 2^1)$

$P_{odd}(n) \ni$ $(9,9,9,9,9,\underbrace{5,5,\ldots,5}_{12},3,3,1,1,1)$

reorder parts $\rightarrow = (9,36,20,40,6,1,2)$.

Some more about formal power series & ordinary generating fn's:

Let's define some specific elements of $\mathbb{C}[[x]]$:

DEF'N $e^x := \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\log(1+x) := \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\lambda \in \mathbb{C}$, $(1+x)^\lambda := \sum_{k\geq 0} \binom{\lambda}{k} x^k$,

where $\binom{\lambda}{k} := \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(k-1))}{k!} \in \mathbb{C}$

(just like $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$ for $n\in\mathbb{N}$)

These satisfy the properties you would expect...

E.g. ① $(1+x)^\lambda (1+x)^\mu = (1+x)^{\lambda+\mu} \in \mathbb{C}[[x]]$

② $e^x e^y = e^{x+y} \in \mathbb{C}[[x,y]]$, ...

③ $\underbrace{e^{\log(1+x)}}_{\text{defined to be}} = 1 + x$, etc...

defined to be $= 1 + \log(1+x) + \frac{(\log(1+x))^2}{2!} + \cdots$

$= 1 + (x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots) + \frac{(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots)^2}{2!} + \cdots$

why does this even converge in $\mathbb{C}[[x]]$?!

General Prop. If $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, and $b_0 = 0$,

then $A(B(x)) := \sum_{n\geq 0} a_n (B(x))^n$ converges in $\mathbb{C}[[x]]$.

Note: $\log(1+x) = x - \frac{x^2}{2} + \cdots$ has 0 constant term.

How to justify ①, ②, ③, etc...? Could do very tedious complex manipulation of coefficients, but instead, trick from analysis;

Thm if $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic for $|z| < R$ (some $R>0$) and f vanishes on $|z|<R$, then $f(z) \equiv 0$, i.e., $a_0 = a_1 = a_2 = \cdots = 0$.
(Standard fact from complex analysis, true under much weaker assumptions)

④ for $n \in \mathbb{N}$, $(1+x)^n = \sum_{n\geq 0} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$

but also $\frac{1}{(1-x)^n} = (1+(-x))^n = \sum_{k\geq 0} \frac{(-n)(-n-1)(-n-2)\cdots(-n-(k-1))}{k!} (-1)^k x^k$

$$= \sum_{k\geq 0} \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} x^k$$

$$= \sum_{k\geq 0} \binom{n+k-1}{k} x^k$$

$\frac{1}{(1-x)^n} = \underset{\text{onion}}{(1+x+x^2+\cdots)}\underset{\text{sesame}}{(1+x+x^2+\cdots)}\cdots\underset{\text{asiago}}{(1+x+x^2+\cdots)}$

$n = \#$ parantheses $= 18$ flavors of bagels

$$**|*||\cdots|*|**$$

$\left(\!\binom{n}{k}\!\right) := \#$ k-element multisets w/ entries in $\{1,2,\ldots,n\}$

$\left(\!\binom{18}{13}\!\right) = \binom{18+13-1}{18}$ (e.g. $\{1,3,3,4,4,4,4\}$ is a 7 elem multiset w/ entries in $\{1,2,3,4\}$)

pf that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$:
"Stars and bars"
stars indicate how many time each element is chosen, bar separate the bins that represent elements.

⑤ $\frac{1}{1-4x} = \sum_{k\geq 0} \binom{-1}{k}(-4x)^k = \sum_{k\geq 0} \underbrace{\binom{1+k-1}{k}}_{=\binom{k}{k}=1} 4^k x^k = \sum_{k\geq 0} 4^k x^k$ ✓

$\frac{1}{1-4x} = (1+(-4x))^{-1}$

but also $\frac{1}{(1-4x)^2} = \sum_{k\geq 0} \underbrace{\binom{2+k-1}{k}}_{\binom{k+1}{k} = k+1} 4^k x^k = \sum_{k\geq 0} (k+1) 4^k x^k$

$\frac{1}{(1-4x)^3} = \sum_{k\geq 0} 4^k x^k \binom{k+2}{2}$, $\frac{1}{(1-4x)^4} = \cdots$ etc.

Useful for extracting coefficients after partial fraction expansions...

⑥ $\frac{1}{\sqrt{1-4x}} = (1-4x)^{-\frac{1}{2}} = \sum_{k\geq 0} \binom{-\frac{1}{2}}{k} (-4)^k x^k$

$= \sum_{k\geq 0} \left(\frac{-\frac{1}{2}\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\cdots\frac{-(2k-1)}{2}}{k!} (-4)^k \right) x^k$

$= \sum_{k\geq 0} \left(\frac{\overset{2^k}{4^k} (1)(3)(5)\cdots(2k-1)}{2^k \quad k!} \right) x^k$

$= \sum_{k\geq 0} \frac{\overbrace{2^k \cdot k!}^{(2)(4)(6)\cdots(2k)} (1)(3)(5)\cdots(2k-1)}{k! \quad k!} x^k = \sum_{k\geq 0} \frac{(2k)!}{k!\,k!} x^k$

interesting... $\longrightarrow$ $= \sum_{k\geq 0} \binom{2k}{k} x^k$

Another tool from calculus for $R[[x]]$ ...

DEF'N for $A(x) = \sum_{n\geq 0} a_n x^n \in R[[x]]$, we define the formal derivative $A'(x) := \sum_{n\geq 1} \underbrace{n a_n}_{\underbrace{a_n + a_n + \cdots + a_n}_{n \text{ times}}} x^{n-1} \in R[[x]]$.

The derivative satisfies usual rules from calculus;

- $(A(x) + B(x))' = A'(x) + B'(x)$
- $(AB)' = A' \cdot B + B'A$
- $\left(\frac{1}{A}\right)' = \frac{-A'}{A^2}$
- $(A(B(x)))' = A'(B(x)) \cdot B'(x)$
- etc...

Quick review of binomials and multinomials

Binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ has several (easy) interpretations

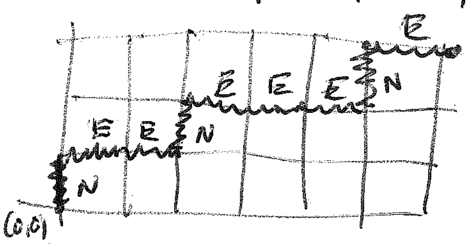
$\left(= \binom{n}{k, n-k} \text{ in multinomial notation}\right)$ $=$ # words with k 1's, $n-k$ 2's

e.g. $\binom{4}{2} = 6 =$ # $\{1122, 1212, 1221, 2121, 2112, 2211\}$

i.e., rearrangements of $\underbrace{11 1 \cdots 1}_{k} \; \underbrace{22 2 \cdots 2}_{n-k}$

$=$ # lattice paths in $\mathbb{Z}^2$ taking east (E) or north (N) steps, from $(0,0)$ to $(k, n-k)$

e.g. $n=9$, $k=6$, $n-k=3$



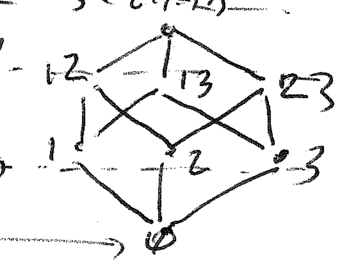
$\longleftrightarrow$ $\binom{9}{6,3}$ E's, N's

1 2 3 4 5 6 7 8 9
N E E N E E E N E

$=$ size of the k^{th} rank in B_n, where B_n is the Boolean lattice of subsets of $\{1, 2, \ldots, n\}$ partially ordered by containment

e.g. $n=3$

rank 3 $\to$ $\{1,2,3\}$
rank 2 $\to$ 12
rank 1 $\to$ 1
rank 0 $\to$ $\emptyset$



$\binom{3}{3} = 1$

$\binom{3}{2} = 3$

$\binom{3}{1} = 3$

$\binom{3}{0} = 1$

Of course, also have binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

multinomials:

How many rearrangements of BANANAS?

i.e., of 3 A's, 1 B, 2 N's, 1 S's! (equiv. of $\overset{1}{A}\overset{2}{A}\overset{3}{A}\overset{4}{B}\overset{5}{N}\overset{6}{N}\overset{7}{S}$)

$\exists$ transitive action of the symmetric group $\mathfrak{S}_7$ on the rearrangements

e.g., permutation $\sigma = \begin{pmatrix}1&2&3&4&5&6&7\\2&1&4&3&6&7&5\end{pmatrix}$ sends $\overset{1\,2\,3\,4\,5\,6\,7}{AAABNNS} \mapsto AABASNN$

The stabilizer of this action is $\underset{\mathfrak{S}_{\{1,2,3\}}}{\underset{\parallel}{\mathfrak{S}_3}} \times \underset{\mathfrak{S}_{\{4\}}}{\underset{\parallel}{\mathfrak{S}_1}} \times \underset{\mathfrak{S}_{\{5,6\}}}{\underset{\parallel}{\mathfrak{S}_2}} \times \underset{\mathfrak{S}_{\{7\}}}{\underset{\parallel}{\mathfrak{S}_1}} \le \mathfrak{S}_7$, $\quad \binom{7}{3,1,2,1}$

So by orbit-stab. thm, $\underset{\#\text{ rearrangements}}{\underset{\parallel}{\text{size of orbit}}} = \dfrac{|\mathfrak{S}_7|}{|\mathfrak{S}_3\times\mathfrak{S}_1\times\mathfrak{S}_2\times\mathfrak{S}_1|} = \dfrac{7!}{3!\,1!\,2!\,1!}$

Multinomial coefficient: $\binom{n}{k_1,k_2,\dots,k_m} = \dfrac{n!}{k_1!\,k_2!\cdots k_m!}$ for $k_1+\cdots+k_m = n$

$= \#$ words w/ k_1 1's, k_2 2's, $\vdots$ k_m m's, i.e. rearrangements of $\underbrace{11\cdots1}_{k_1}\underbrace{22\cdots2}_{k_2}\cdots\underbrace{m\cdots m}_{k_m}$

$= \#$ lattice paths in $\mathbb{Z}^m$ taking steps $e_1, e_2, \dots, e_m$ (standard basis) from $0 = (0,0,\dots,0)$ to $(k_1, k_2, \dots, k_m)$

(same correspondence between words + paths as binomials)

$= \#$ chains (or flags) in B_n of subsets

$\phi = S_0 \subsetneq S_{k_1} \subset S_{k_1+k_2} \subset \cdots \subset S_{k_1+k_2+\cdots+k_{m-1}} \subset S_n = \underset{[n]}{\underset{\parallel}{\{1,2,\dots,n\}}}$

passing through ranks $0, k_1, k_1+k_2, \dots, k_1+k_2+\cdots+k_{m-1}, n$.

e.g. for $\binom{7}{3,1,2,1}$ have $2131314 \leftrightarrow \phi \overbrace{\subset S_3}^{3} \overbrace{\subset S_4}^{1} \overbrace{\subset S_6}^{2} \overbrace{\subset [7]}^{1}$.

$S_3 = \{2,4,6\}$, $S_4 = \{1,2,4,6\}$, $S_6 = \{1,2,3,4,5,6\}$

Note: $\binom{n}{k_1,k_2,\dots,k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-(k_1+k_2)}{k_3}\cdots\binom{n-(k_1+\cdots+k_{m-2})}{k_{m-1}}\binom{k_m}{k_m}$

Also, **multinomial theorem**: $(x_1+x_2+\cdots+x_m)^n = \displaystyle\sum_{\substack{(k_1,k_2,\dots,k_m)\\ k_1+\cdots+k_m=n}} \binom{n}{k_1,k_2,\dots,k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$.