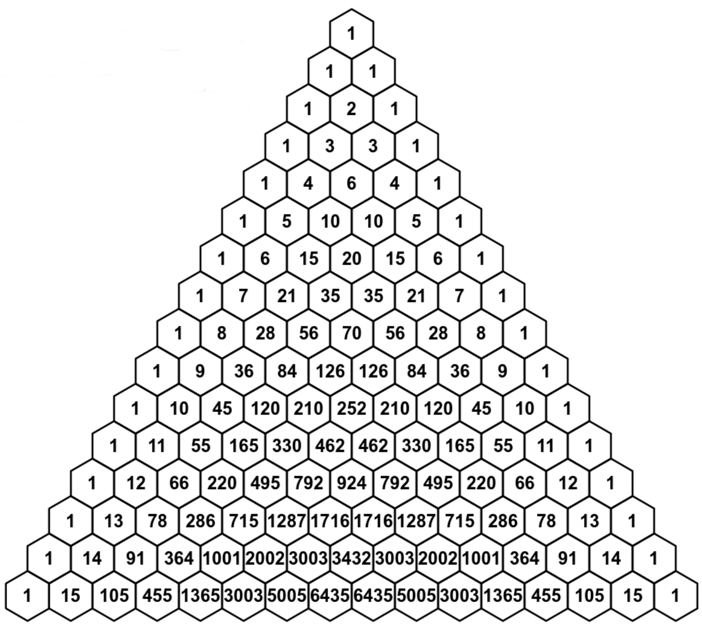
Binomial coefficients and Pascal's triangle, UMTYMP Advanced Topics, Fall 2020

We proved the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ are the *binomial coefficients*. The binomial coefficients fit into Pascal's Triangle:



1. Prove $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ using the Binomial Theorem. (**Hint**: derivatives!)
2. Prove $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ combinatorially.
3. Prove $\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$ combinatorially.
4. Deduce that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ from the previous item.
5. Prove that $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ combinatorially.
6. Fill in the odd numbers in the above Pascal's triangle. Do recognize the image?
7. What about the binomial coefficients $= 1 \mod 3$? (You can look up **Lucas's Theorem** to learn more about arithmetic properties of binomial coefficients.)