Catalan numbers,
UMTYMP Advanced Topics, Fall 2020

The *Catalan number* sequence

$$C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, \ldots,$$

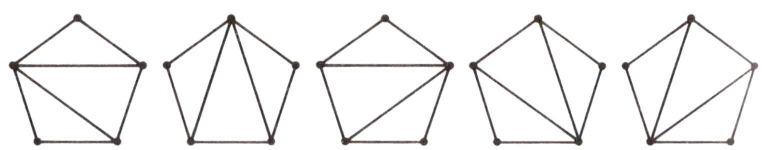
with general formula $C_n = \frac{1}{n+1}\binom{2n}{n}$, and generating function $C(x) := \sum_{n=0} C_n x^n$ given by $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, is ubiquitous in combinatorics.

The Catalan numbers satisfy the *fundamental recurrence*

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

For each of the following possible definitions of C_n, explain why the fundamental recurrence holds:

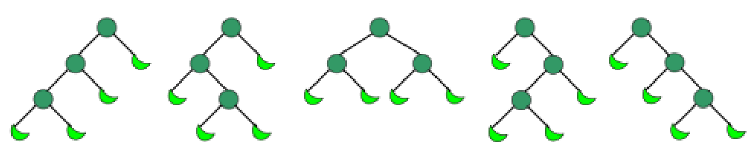
1. C_n := number of triangulations of an $n+2$-gon; the case $C_3 = 5$ corresponds to



2. C_n := number of words of length $2n$ with n X's and n Y's such that every initial segment has at least as many X's as Y's (these are called *Dyck words*); the case $C_3 = 5$ corresponds to

   XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY

3. C_n := number of *binary trees* (each node either has either two children: a left and a right child; or has no children and is a "leaf') with $n+1$ leaves; the case $C_3 = 5$ corresponds to:



For each of the following possible definitions of C_n, explain a bijection to one of the above definitions:

4. $C_n :=$ number of lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ and $(1,0)$ staying on or above diagonal $y = x$ (these are called *Dyck paths*); case $C_3 = 5$:

5. $C_n :=$ number of ways to fill a $2 \times n$ rectangle with the numbers $1, 2, ..., 2n$ increasing in rows and columns; case $C_3 = 5$:

1	2	3
4	5	6

1	2	4
3	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

6. $C_n :=$ number of ways to completely parenthesize $n+1$ different factors; case $C_3 = 5$:

(((ab)c)d) ((a(bc))d) ((ab)(cd)) (a((bc)d)) (a(b(cd)))

7. $C_n :=$ number of ways for $2n$ people seated at a circular table to shake hands without crossing; case $C_3 = 5$: