Generating functions,
UMTYMP Advanced Topics, Fall 2020

Let $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, ...$ with $F_n = F_{n-1} + F_{n-2}$ be the Fibonacci number sequence. We saw that the generating function $F(x) := \sum_{n=0}^{\infty} F_0 x^n$ for the Fibonacci numbers is $F(x) = \frac{x}{1-x-x^2}$ and, via partial fraction decomposition, $F(x) = \frac{1/\sqrt{5}}{1-x(1+\sqrt{5})/2} - \frac{1/\sqrt{5}}{1-x(1-\sqrt{5})/2}$. Using the geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$, we got $F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$ and thus $F_n \approx \frac{1}{\sqrt{5}}\phi^n$ where $\phi = (\frac{1+\sqrt{5}}{2}) \approx 1.618...$ is the golden ratio.

1. Consider the sequence $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 3, J_4 = 5, ...$ with $J_n = J_{n-1} + 2J_{n-2}$. Form the generating function $J(x) := \sum_{n=0}^{\infty} J_n x^n$. Show that $J(x) = \frac{x}{1-x-2x^2}$.
2. Use partial fractions to show that $J(x) = \frac{1/3}{1-2x} - \frac{1/3}{1+x}$.
3. Conclude that $J_n = \frac{2^n - (-1)^n}{3}$, so that $J_n \approx \frac{1}{3}2^n$.

For $n \in \mathbb{N}$, the generating function for the binomial coefficients $\binom{n}{k}$ is $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$ (by the binomial theorem). Generalizing this, for any real number $n \in \mathbb{R}$, define $\binom{n}{k} := \frac{n(n-1)(n-2)...(n-(k-1))}{k!}$. The *generalized binomial theorem* says that for any $n \in \mathbb{R}$ we have $\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$.

4. Show that $(1+x)^{-4} = \sum_{k=0}^{\infty} (-1)^k \binom{4+k-1}{k} x^k$ and thus that $\left(\frac{1}{1-x}\right)^4 = \sum_{k=0}^{\infty} \binom{4+k-1}{k} x^k$.
5. Explain how the previous result relates to the "giving pennies to kids"/"choosing bagels" problem (hint: this case = 4 flavors of bagels).
6. Show that $(1+x)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \frac{1\cdot 3\cdot 5 \cdots (2k-1)}{2^k k!} x^k$ and thus that $\frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$ is the *central binomial coefficient* generating function.
7. Use the previous result to show that for all $k \in \mathbb{N}$, $4^k = \sum_{j=0}^{k} \binom{2j}{j}\binom{2(k-j)}{k-j}$.