Fall 2020 9/8

Math 4990: UMTYMP Advanced Topics
Combinatorics!

Instructor: Sam Hopkins

Plan for today:

- Logistics ✓
- Introductions ✓
- Overview of course
- Two basic techniques:
  - Induction
  - Pigeonhole principle
- Try some group work

Class logistics:

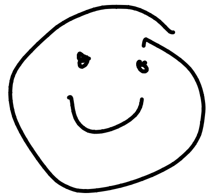
- Class is 4-6 pm CDT Tuesdays, on Zoom
- You should all be part of the Course on Canvas
- Main webpage for the class:

umn.edu/~shopkins/classes/UMTYMP.html

- Text: Bona's "Walk through Combinatorics"
- Assignments (all take-home): 5 HW's, 2 Midterms, 1 Final
- Try to divide 2 hr class time into lecture + group work

- We'll develop proof-writing skills

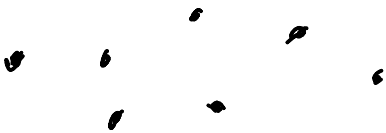
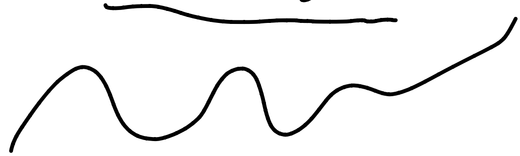
Introductions!



- Say who you are.
- Say where you are.
- Say one thing you've been doing to stay grounded during quarantine.

Overview of the course:

Combinatorics is study of $\boxed{\text{discrete}}$ structures.

Discrete	Continuous
	
finite	infinite
integers $\mathbb{Z}$	real numbers $\mathbb{R}$
$\ldots, -2, -1, 0, 1, 2, 3 \ldots$	
algebra (ish)	Calculus
quantum physics (ish),	classical physics
computer science	

Specifically, we'll discuss...

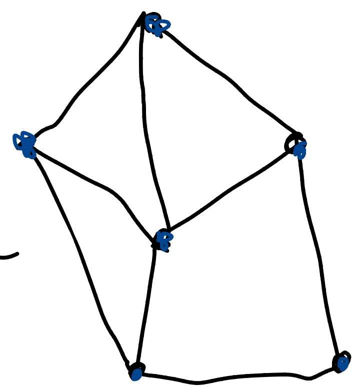
- enumeration,

a.k.a counting:

- how many orderings of n things?
- how many subcollections? etc.

- graph theory

graphs = digrammatic representation of a network



look for structures in such a network

Maybe...

- algorithms + optimization

Today... we'll go over Ch's 1+2 of Bona which review 2 important techniques for proving things in the discrete world.

#2: Induction (Chapter 2)

Context:

$\boxed{\text{Have a statement } P(n) \text{ depending a (nonnegative integer) parameter } n.}$

If you can:

- Prove $P(n_0)$ for some base case n_0 (usually, $n_0 = 0$ or 1)
- Prove $P(n)$ implies $P(n+1)$, (inductive step)

then you've proved $P(n)$ for all $n \geq n_0$!

Examples:

1) Prove that

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Pf: Base case: $n=1$:

$$1=1^2 \overset{?}{=} \frac{1(1+1)(2+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1 \checkmark$$

Assume case $n=k$, i.e.,

$$1^2+2^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$$

then,

$$1^2+2^2+\cdots+k^2+(k+1)^2$$

$$\text{L} \stackrel{?}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$=$ ... algebraic manipulation ...

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Case $n=k+1$ of statement

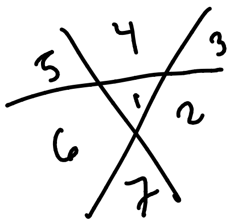
So by induction, we're done. $\boxtimes$

2) If we draw n lines in the plane in "generic position"

(no parallel lines, no 3 intersect in 1 point)

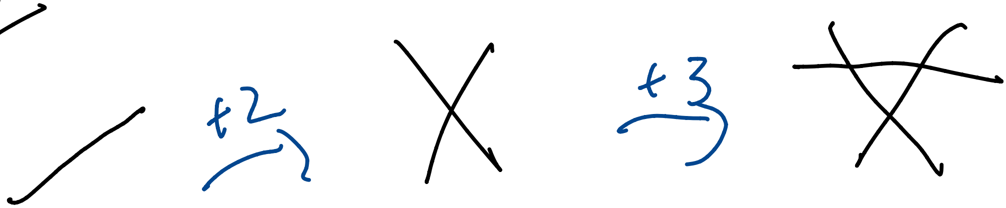
how many regions do we get?

e.g.,



3 lines ⇒ 7 regions

pf:



$n=1$ 2 regions

$n=2$ 4 regions

$n=3$ 7 regions

+1

$n=0$ 1 region

pattern: nth line adds n regions

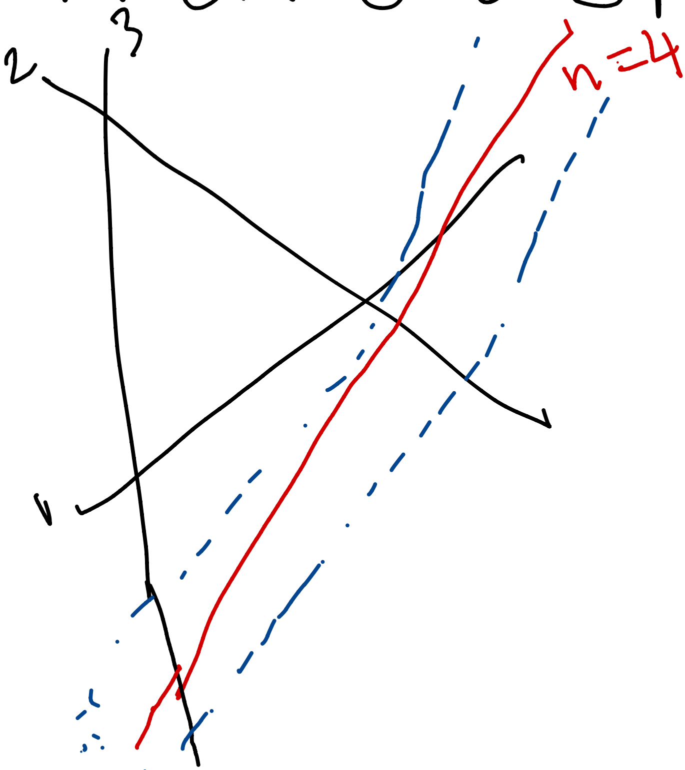
So # regions =

$1 + 1 + 2 + 3 + \cdots + n = 1 + \frac{n(n+1)}{2}$

know

Base case, $n = 0$ ✓

Inductive step:



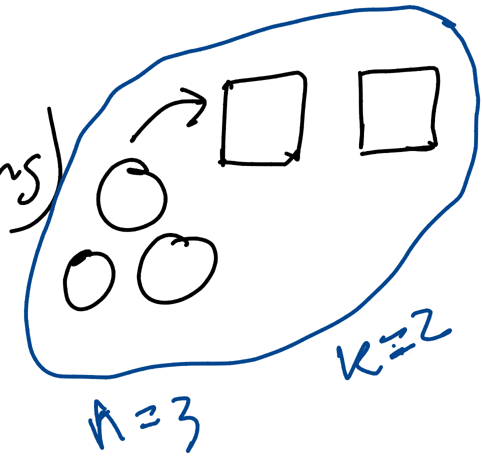
← adding nth line gives n new regions, since it passes thru n regions, and cuts them into 2, which gives n new regions. ∎

#2 Pigeonhole Principle

(Chapter 1)

Context:

Putting balls (or pigeons) into boxes.



—

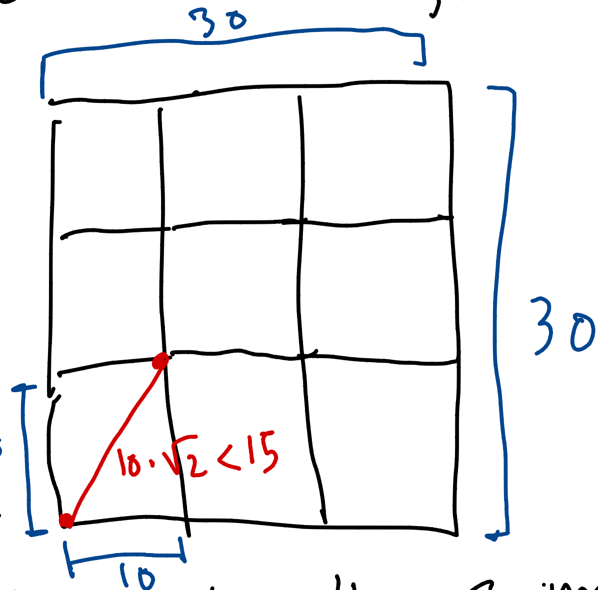
Proposition

If we put n balls into k boxes, where $n > k$, then at least one box has ≥ 2 balls in it.

Pf: Assume not. Then every box has 0 or 1 pigeons in it. Then the most balls we could have is k, but $n > k$, a contradiction. ∎

Ex 1) Ten cows are in a 30×30 ft square pen. Show that there must be 2 cows within 15 ft. of each other.

Pf: Hint:



at least

PP $\Rightarrow$ 2 cows are in the same square

furthest distance they could be is at diagonals

$\Rightarrow \; \leq 10\sqrt{2}$ ft. < 15 ft. $\blacksquare$

2) Some number in the sequence

$9, \ 99, \ 999, \ 9999, \ \ldots$

is divisible by 2019.

Pf. Hint:

$$\begin{array}{rl} 9999999 & \equiv k \mod 2019 \\ -\ 999 & \equiv k \mod 2019 \\ \hline 9999000 & \equiv 0 \mod 2019 \end{array}$$

Since there are only 2019 remainders when we divide by 2019, two of the #s in our seq. have same remainder.

$\rightarrow = 9999 \times 10^3$ is divisible by 2019.

But $\gcd(2019, 10^k) = 1$ (they're coprime),

so if $\underbrace{9\cdots9}\times 10^k$ is divisible by 2019,

means $9\cdots9$ is divisible by 2019. $\boxed{}$

Now let's take a break! . . .

and then come back and work

in groups on some problems

related to induction

and the pigeonhole principle.