Math 4990: Graph coloring

11/17

Half of Ch. 11

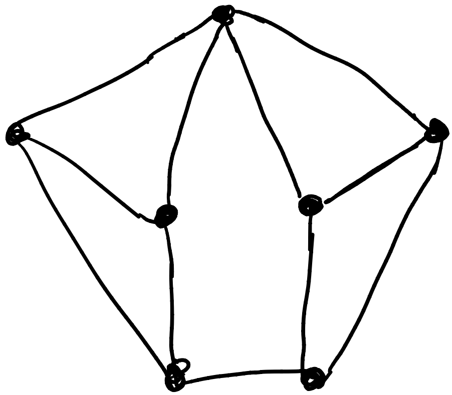
Reminders: • HW#4 should be graded + returned soon, if not already.

• Midterm #2 is due today.

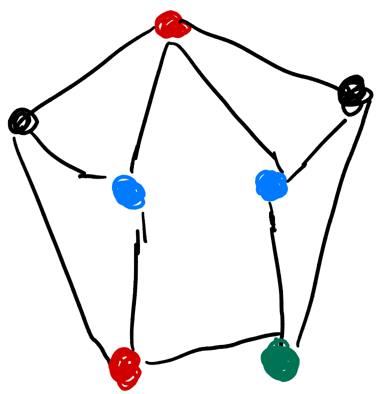
Consider the following "real world problem": there are a number of radio towers in a region, and four possible frequencies they could broadcast at; but towers that are close to one another should not be given the same frequency b/c they might interfere with each other; how can you find a valid assignment of frequencies?

Can rephrase problem in graph theory terminology. Lets draw a graph where the vertices represent the towers, with two vertices joined by an edge if the towers are close.

For example:



Then an assignment of frequencies to the towers is the same as a coloring of the vertices of the graph with four colors s.t. adjacent vertices have different colors:



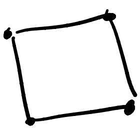
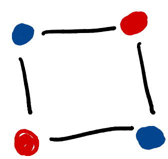
These are the kinds of problems we will study today. (The 'coloring' term. comes from maps, which we'll discuss later...)

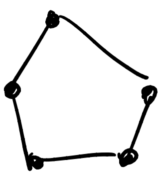
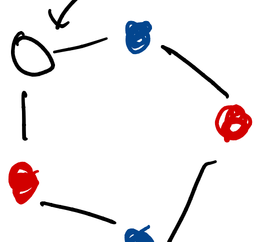
Def'n: A (proper) K-coloring of a graph G is an assignment of K colors to its vertices so that adjacent vertices are colored differently.

[TODAY: ALL GRAPHS ARE SIMPLE]

Most basic question we can ask about graph coloring is: how few colors do we need?

Def'n The chromatic number $\chi(G)$ of a graph G is the minimum K s.t. G has a K-coloring.

e.g. $\chi\Big($  $\Big) = 2$ since 

$\chi\Big($  $\Big) = 3$ since 

can't 2-color!

Let's think about graphs G w/ $\chi(G) = K$ for small values of K.

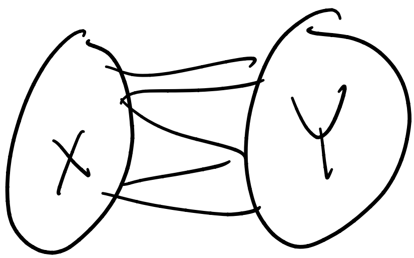
$\chi(G)=1 \Rightarrow G$ has no edges ✓

$\chi(G)=2 \cdots$ this is a very interesting condition!

Def'n. A graph G w/ $\chi(G)=2$ is called bipartite.

Why "bipartite"? B/c ...

Prop. G is bipartite $\Leftrightarrow$ G can be partitioned into 2 vertex subsets X, Y such that no edges within X + Y, only between:

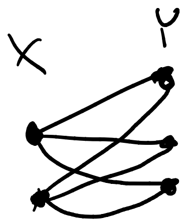


Pf: Think about X and Y as the 2 color classes! ∎

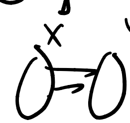
Bipartite graphs are a very important class of graphs in graph theory.

Q: What's the most edges a bipartite graph w/ n vertices can have?

Def'n The complete bipartite graph $K_{a,b}$ has one part X of size a, one part Y of size b, and all edges between X and Y.

e.g. $K_{2,3} =$ 

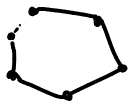
Prop. The most edges a bipartite graph on n vertices can have is $\left(\frac{n}{2}\right)^2$ (n even) or $\frac{n+1}{2} \cdot \frac{n-1}{2}$ (n odd).

Pf: Take some graph achieving max. It looks like 

. If missing some edges btween X + Y, could add them. So it must be complete. Then # edges of $K_{a,b}$ maximized when $a = b$ (or $a = b-1$). (for fixed $a+b$.) ∎

Q: Can we characterize bipartite graphs in any useful way?

We saw before why for a cycle graph

$C_n =$  on n vertices we have

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

This basically determines bipartiteness:

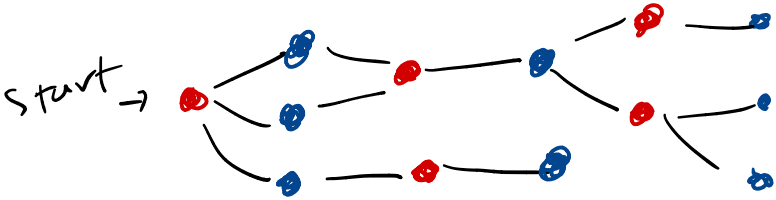
Thm A graph G is bipartite $\Leftrightarrow$ it has no odd cycles.

pf: If G has an odd cycle, then certainly we cannot 2-color G, since we can't even 2-color that cycle.

Now assume G has no odd cycles. We want to show we can 2-color it.

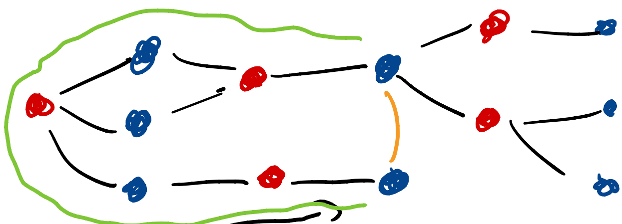
So... let's just try. Start anywhere,

and color that vertex red. Then color its neighbors blue. Then color their neighbors red. And so on... We make a "tree": ← not really a tree but looks like this



(tree goes this way →)

Can assume G is connected, and we've colored it all this way. Why is coloring proper? Suppose not, e.g. $\exists$ edge b/tween same color verts:



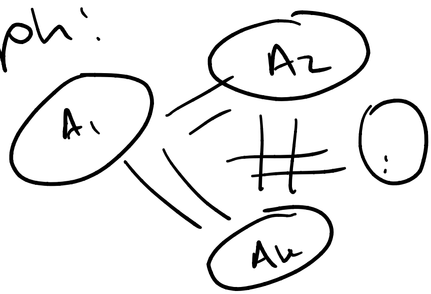
1st note: can only happen b/tween verts at same level b.c. otherwise would've colored one earlier. Then, find paths in tree back to common ancestor in tree; together w/ edge between them, this gives a cycle of odd length, Contradiction. So indeed the 2-coloring is proper. $\blacksquare$

Okay, so what about graphs w/ $\chi(G)=3$? Or 4? Or more? Understanding coloring for graphs w/ $\chi(G)\geq 3$ is much harder than for bipartite graphs.

Basic issue: We saw in proof above that a 2-coloring, when it exists, is (basically) unique. But for k-colorings, $k\geq 3$, this is far from true: there are many choices.

Of course, there are still things we can say about k-colorings in general:

The complete k-partite graph $K_{a_1,a_2,\ldots,a_k}$ is graph:



($|A_i|=a_i$, no edges in A_i, all other edges)

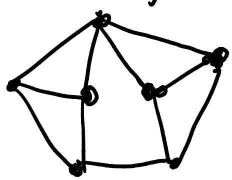
It has $\chi(K_{a_1,\ldots,a_k})=k$, and it maximizes #edges when the a_i roughly equal.

Prop: If G contains a subgraph $\simeq K_d$, (complete graph) then $\chi(G) \geq d$.

Pf: K_d clearly has $\chi(K_d) = d$. $\blacksquare$

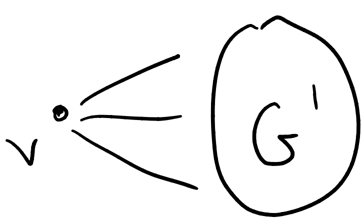
Rmk: Converse of this prop is not true!

We've already seen odd cycles.

Also $\chi\Bigg($  $\Bigg) = 4$, but it has no K_4.

Prop: Let $\Delta(G)$ denote the maximum degree of G. Then $\chi(G) \leq \Delta(G) + 1$.

Pf: By induction. Let v be any vertex of G, and $G' = G - v$:



By induction, we can color G w/ at most $d = \Delta(G)+1$ colors. Also, $\deg(v) \leq \Delta(G) = d-1$, so among neighbors of v, at most $d-1$ colors are used, leaving at least one for v. $\blacksquare$

Rmk: Again, the bound in this prop. can be far from the truth (see worksheet).

Rmk: Deciding if a graph G has $\chi(G)=3$ is a hard problem, in the computer science sense of hard, just like some other problems we've seen: existence of Hamilton. cycle, etc.

Now let's take a break!...

And when we come back let's do some worksheet problems about coloring in breakout groups.