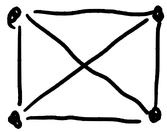
Math 4990: Planar graphs

12/1

Ch. 12 of Bóna

Reminder: HW #5 is due today.

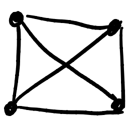
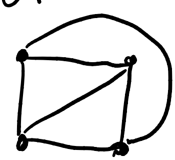
When we've been drawing graphs on paper, we haven't cared whether the edges cross:

e.g. $K_4 =$ 

Today we'll think about disallowing crossings.

Def'n A graph that can be drawn in the plane w/out edges crossing (except at vertices) is called a planar graph. Such a drawing is called a planar embedding.

Note: Even if some drawing of G might have a crossing, another might not:

e.g. $K_4 =$  $=$  ← planar embedding

Remember: The edges don't have to be straight, they can be curves.

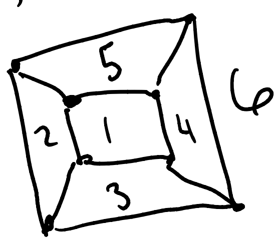
Q for today: What graphs are planar? What properties do planar graphs have?

Note: multiple edges and loops don't

affect planarity, so we'll implicitly only discuss simple graphs.

Faces

A planar embedding divides the plane into certain regions, which we call faces:



In the above example, there are 6 faces. Notice that there is always an unbounded face called the outer face.

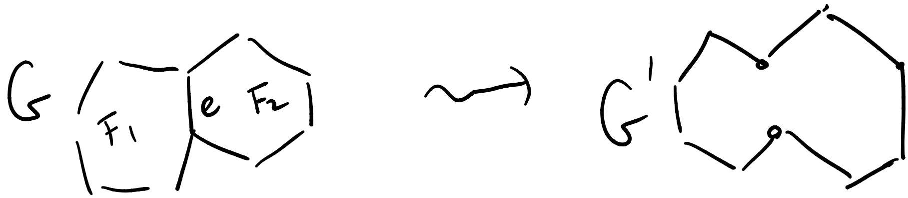
Thm (Euler's formula) For G connected + planar,

$$V + F = E + 2,$$

where $V = \#$ vertices, $E = \#$ edges, $F = \#$ faces.

e.g. above... $8 + 6 = 12 + 2$ ✓

Pf: The proof is by induction on #edges of G.
Suppose there's an edge e of G whose removal does not disconnect G. Then e belongs to a cycle of G, so it separates two faces:



Let $G' := G - e$. Then G' has one less edge, one less face, and same number of vertices as G, so if

$$V + F = E + 2$$

holds for G', it holds for G.

Now suppose there is no edge e whose removal disconnects G. Then G is a tree! A tree has one face, and we know that if it has n vertices then it has $n-1$ edges. So Euler's formula:

$$n + 1 = (n-1) + 2. \checkmark$$

$\blacksquare$

Euler's formula is very powerful, and for instance restricts # of edges planar graph can have.

Cor Let G be a (simple) planar graph. Then,

$$\#E(G) \le 3 \cdot \#V(G) - 6.$$

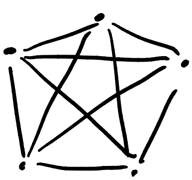
Pf: Each edge is in $\le$ ~~exactly~~ 2 faces, and each face has ≥ 3 edges (△ or □ or ...)

So $$\#E(G) \ge \frac{3}{2}\#F(G) \qquad (*)$$

Euler $\Rightarrow$ $$\#F(G) + \#V(G) = \#E(G) + 2 \qquad (**)$$

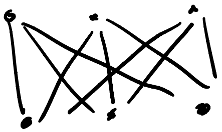
$(*)+(**) \Rightarrow$
$$\frac{2}{3}\#E(G) + \#V(G) \ge \#E(G) + 2$$
$$\#V(G) - 2 \ge \frac{1}{3}\#E(G)$$
$$3\#V(G) - 6 \ge \#E(G). \checkmark$$

e.g. K_5 is not planar, since it has 5 vertices and 10 edges, but



$$10 \not\le 3 \cdot 5 - 6 = 15 - 6 = 9.$$

e.g. Complete bipartite graph $K_{3,3}$ is not planar



It has 6 vertices, 9 edges

$9 \leq 3 \cdot 6 - 6 \cdots$ so?

But in bipartite graph, min. cycle size is 4!

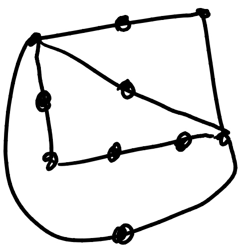
So planar bipartite graph $\#E(G) \leq 2\#V(G) - 4$.

and $9 \not\leq 2 \cdot 6 - 4 = 8$. //

Def'n A subdivision of G is graph obtained by doing

repeatedly,

e.g.



is a subdivision of K_4.

Easy prop: G is planar $\Leftrightarrow$ subdivision of G is planar.

Also easy: G is planar $\Rightarrow$ any subgraph of G is planar.

Thm (Kuratowski's Thm)

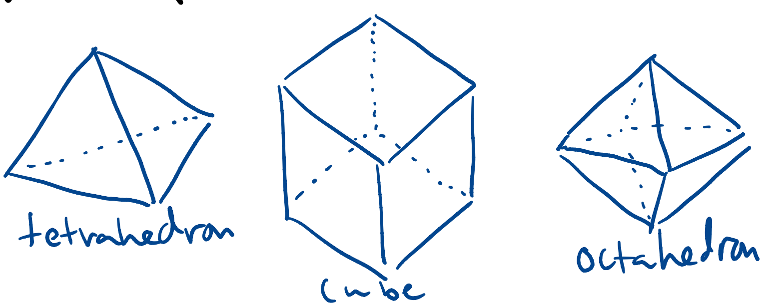
G is planar $\iff$ no subgraph of G is a subdivision of K_5 or $K_{3,3}$.

Pf: Beyond this class . . . .

Polyhedra

Def'n A **convex polyhedron** is a 3D shape made up of flat things (vertices, edges, faces) that doesn't "go in" anywhere.

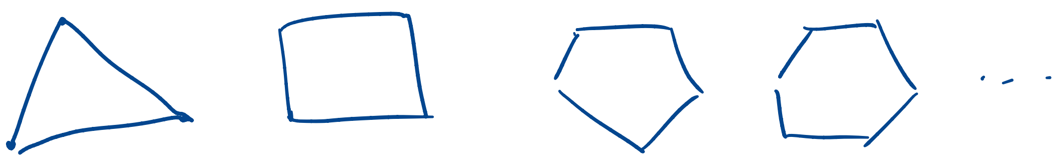
You've probably seen the **Platonic solids**:



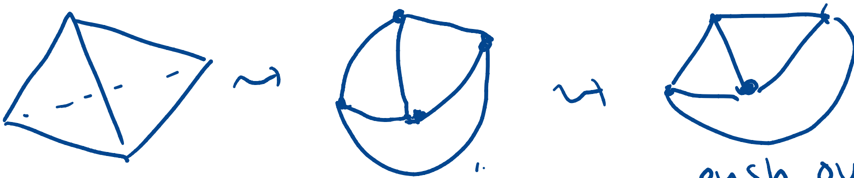
12 faces — dodeca-hedron

20 faces — icosa-hedron

Compare to **convex polygons**:

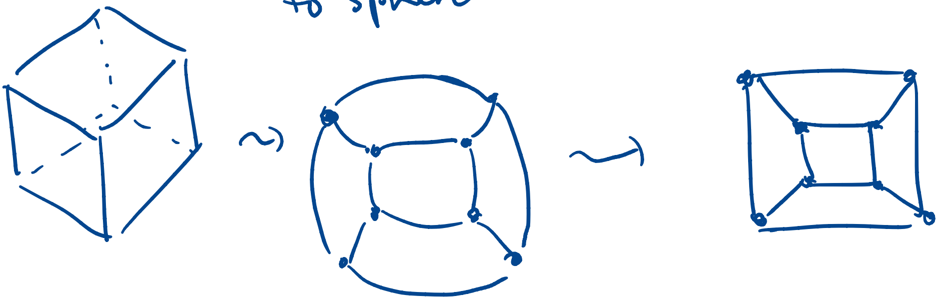


Convex polyhedra $\rightsquigarrow$ planar graphs



"blow up" to sphere

push out one face to outer face



Cor For a convex polyhedron P,
$$\#V(P) + \#F(P) = \#E(P) + 2.$$
✓

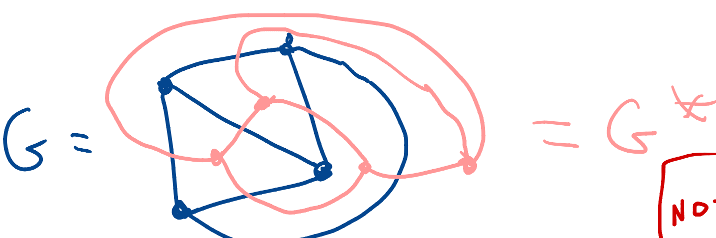
Dual graphs

To any planar graph G, can associate a **dual graph** G^* where

by drawing "crossing edges."

$$\text{vertices}(G^*) = \text{faces}(G)$$
$$\text{faces}(G^*) = \text{vert's}(G)$$
$$\text{edges}(G^*) = \text{edges}(G)$$

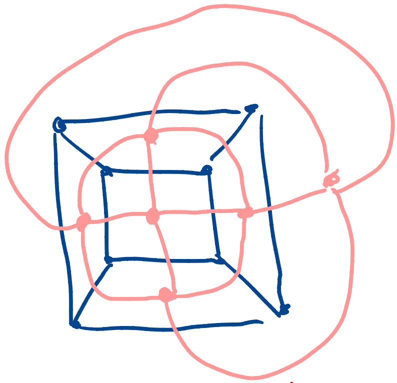
e.g.



$\boxed{\text{NOTE: } (G^*)^* = G}$

Note that in this example, $G \simeq K_4 \simeq G^*$, so K_4 is self-dual.

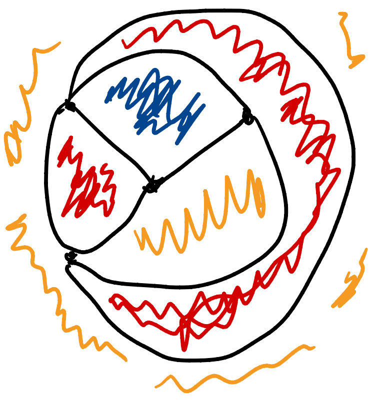
e.g. G cube =



$= G^*$ octahedron

duality of polyhedra! (known to ancient Greeks)

Coloring planar graphs is a big topic.
Goes back to map coloring:



Note: Properly coloring the faces of G = properly coloring the vertices of G^*.

So stick to vertex coloring...

Thm Every planar graph G has chromatic # ≤ 6.

Pf: First we need a lemma:

Lemma Every planar graph G has a vertex of degree ≤ 5.

Pf: Assume all deg's ≥ 6. Then

$$2 \cdot E = \sum \deg \geq 6 \cdot V \Rightarrow E \geq 3 \cdot V,$$

but we know that $E \leq 3V - 6$. ☒

So let v be vertex of G w/ $\deg(v) \leq 5$.
Let $G' := G - v$. G' is still planar, so by induction can 6-color G'. And can extend to 6 coloring of G since v has at most 5 neighbors so there's at least one color left for it. ☒

Little bit more work:

Thm $\chi(G) \leq 5$ $\forall$ planar graphs G.

Lotsa lotsa lotsa more work!:

Thm (4 color Theorem)
$\chi(G) \leq 4$ $\forall$ planar graphs G.

only known proof uses huge computer check!

Now let's take a break...

and when we come back, work on planar graph stuff on the worksheet in breakout groups!