Math 4990: Basic enumeration problems (Ch. 3) 9/15

Before we discuss math today, a few house-keeping items:

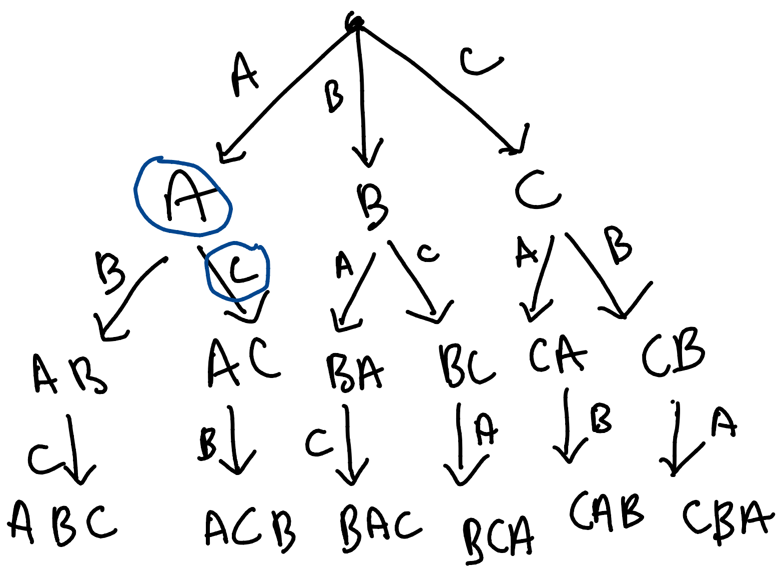
- Web stuff ok? (videos...)
- Office hours: right now, by appointment 1-on-1, but could do a specific time if wanted

- HW #1 posted, due in 1 week.

Last class we reviewed some basic proof techniques. Today we'll properly begin Combinatorics by considering basic counting problems.

Problem: How many ways to order the letters A, B, C are there?

Solution: Think of picking letters one-by-one.



3 choices for 1^{st} letter $\times$ 2 choices for 2^{nd} $\times$ 1 choice for 3^{rd} $= 6$ ways to order ABC

More generally,

Thm The # of ways to order n distinct objects (i.e., # of permutations of these objects)

$$= n \times (n-1) \times \cdots \times 2 \times 1 =: n!$$

"n factorial"

The proof is the same as case $n=3$: think of choosing 1-by-1.

Here we are implicitly using the

multiplication principle

which (roughly) says that if you can construct something in steps so that the # of choices is always the same at each step, then total # of choices $= \prod$ # of choices at each step.

2^{nd} Problem: How many ways to rearrange letters A A A B B C C C C?

Solution:

Hint: $A_1 A_2 A_3 B_1 B_2 C_1 C_2 C_3 C_4$

eveyone said 9!/3!2!4!

If we label the letters like this, then we get 9! permutations

But 3! ways to assign 1,2,3 as subcripts to the A's, and similarly for B's and C's, which we should divie by

Can generalize previous problem to...

Thm If we have k different types of objects, and $a_i :=$ # objects of type i, for $i=1,\dots,k$, with $n := a_1 + a_2 + \dots + a_k$ objects in total, then the # of ways to order these is

$$\boxed{\frac{n!}{a_1!\, a_2! \cdots a_k!}}$$

(rearrangements/anagrams)

Again the proof is the same as in the previous example.

Note if $a_i = 1$ for all i, get permutations from before.

3rd Problem: How many words (or strings) of length K from an alphabet of size n are there?

Note: unlike w/ permutations, now we can repeat letters as much as we want. E.g., $\boxed{00102312}$ is a word of length 8 in the alphabet $\boxed{\{0,1,2,3,4\}}$

Solution: $n \times n \times \cdots \times n = n^K$, since we have n independent choices for each of the K letters.

Example: if any 10 digits give a phone #, then there are

$10^{10} = 10$ trillion phone #'s.

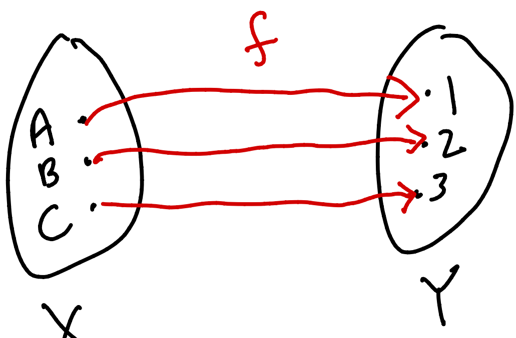
Digression: Bijections

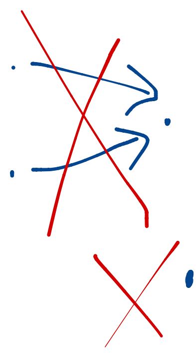
Def'n A bijection between two sets X and Y is a function $f: X \to Y$

s.t. 1) if $f(a)=f(b)$, then $a=b$, for all $a,b \in X$ (injective)

2) for all $y \in Y$, there is some $x \in X$ for which $f(x)=y$ (surjective)

Picture:





Observation:

If there's a bijection from X to Y, then X and Y have the same size.

Example:

Prop.: The # of subsets of $[n] := \{1, 2, \dots, n\}$ is 2^n.

Pf: We could easily use induction,

Instead, let's create a bijection

$$f : \{\text{subsets of } [n]\} \to \left\{\begin{array}{l}\text{words of length } n \\ \text{in alph. } \{0,1\}\end{array}\right\}$$

$\# = 2^n$

Ideas for f: ???

if 1 is in the subset, then 1st letter of word is 1, otherwise 0
if 2 is in the subset, then 2nd letter of word is 1, and so on

e.g. n = 7 and S={1,5,6}, then f(S)=1000110

$\blacksquare$

Problem 4: How many words of length k from alphabet of size n if we can use each letter at most once.

Solution

$$\underbrace{n \times (n-1) \times \cdots \times (n-k+1)}_{k \text{ terms}} = \frac{n!}{(n-k)!}$$

multiplication principle!

because...

Problem 5 (most important?):

How many size k subsets of $[n]$ are there?

Solution: $\dfrac{n!}{(n-k)!\,k!}$, because there are $n!/(n-k)!$ length k strings w/ each letter at most once, and these each determine a subset, but there are $k!$ ways to reorder the subset. E.g.,

$$\begin{matrix} 358 & 583 \\ 385 & 835 \\ 538 & 853 \end{matrix} \longrightarrow \{3,5,8\}$$

$\boxtimes$

Q: Does anyone see another proof?

Hint: What about the 01-word subset bijection?

bijection f from before

rearrangements of (n-k) 0's and k 1's = n!/k!(n-k)!

✓

These #s are so important, we give them a special symbol

$$\binom{n}{k} := \frac{n!}{(n-k)!\,k!}$$ "n choose k"

and name (binomial coefficients)

will explain name next class.

In the book, also consider...

Problem 6: How many multisubsets of size k of [n] are there?

(multi means we can choose element more than once.)

I recommend you read book to see why solution is $\boxed{\binom{n+k-1}{k}}$.

There are four flavors of bagels:
plain, everything, onion, cinnamon raisin

You want to select 13 bagels

How many ways to do it?

Stars and bars!

(plain) | (everything) | (onion) | (cinnamon raisin)

****|***| | ****** = 4 plain , 3 everything, 0 onion, 6 CR

#ways = 16 choose 13 = 16!/(13! * 3!) #= (n+k-1) choose k

Now let's practice counting in the context of poker hand probabilities!

If X is set of possible outcomes, and $A \subseteq X$ is some subset, then the probability our outcome is in A is just

$$\Pr(A) = \frac{\#A}{\#X}$$

("uniform distribution")