Math 4990: Binomial Theorem and Pascal's Triangle (Ch. 4) 9/22

Reminder: HW #1 due today.

Please let me know ASAP if you're having any trouble uploading it, etc.

Last class, we introduced the

binomial coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \#\ k\text{-element subsets of } [n] = \{1, 2, \ldots, n\}.$$

But we didn't explain the name, which comes from this theorem in algebra:

Thm (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

E.g.,

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$= \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}x^0y^3.$$

Pf: Think of expanding

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}^{n \text{ terms}}$$

To write this as a sum of $x^i y^j$'s, for each $(x+y)$ we 'pick' either the x or y.

To get a term of $x^{n-k}y^k$, we must choose the y from exactly k of the $(x+y)$'s and the x from the others.

There are exactly $\binom{n}{k}$ ways to choose which k out of the n $(x+y)$'s we select the y's from. $\blacksquare$

Some consequences of binomial theorem...

Prop. 1 $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Pf: In binomial thm,
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$
Set $x := 1$ and $y := 1$. $\square$

Prop. 2 $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Pf: In $(x+y)^n = \sum \binom{n}{k} x^{n-k} y^k$,
Set $x := 1$ and $y := -1$. We have
$$0^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \longleftarrow \text{why?}$$
$\square$

These are algebraic proofs. It's nice to also try to find combinatorial/bijective proofs.

Combinatorial pf of 1:

$2^n = \#$ subsets of $[n]$

$\sum_{k=0}^{n} \binom{n}{k} = \#$ 0-subsets of $[n]$ $+ \#$ 1 subsets of $[n]$ $+ \cdots +$ $\#$ n-subsets of $[n]$

$=\#$ subsets of $[n]$. ▨

Combinatorial pf of 2: For $n>0$, Identity is equivalent to ← why?

$E :=$ # even sized subsets of $[n]$ $=$ # odd sized subsets of $[n]$ $=: O$

We can define a bijection $f: E \to O$, by
$$f(S) = \begin{cases} S \cup \{1\} & \text{if } 1 \notin S, \\ S \setminus \{1\} & \text{if } 1 \in S. \end{cases}$$

(Why does this work?) ▨

On the worksheet for today, you'll look at more binomial coefficient identities.

Fun to try to find both algebraic and combinatorial proofs of these identities.

The book also discusses 2 generalizations of the binomial theorem...

<u>Thm</u> (Multinomial Thm)

$$(x_1 + x_2 + \cdots x_k)^n = \sum_{a_1 + \cdots + a_k = n} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k},$$

where $\binom{n}{a_1, \ldots, a_k} := \dfrac{n!}{a_1! a_2! \cdots a_k!}$ ← Same as anagram #'s from last class.

are the multinomial coeff's.

For any real number m, can define

$$\binom{m}{k} := \frac{m(m-1)\cdots(m-k+1)}{k!}$$ e.g., $\binom{\pi}{2} = \frac{\pi(\pi-1)}{2}$

<u>Thm</u> $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$ for any real number m.

↑ This is a power series! We'll discuss later...

Binomial thm suggests looking at #'s

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n-1} \quad \binom{n}{n}$$

in a row. Actually, can fit all $\binom{n}{k}$'s nicely into an array called Pascal's triangle:

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & & \vdots & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

It's easy to fill out Pascal's triangle thanks to the fundamental recurrence:

Prop. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pf: $\binom{n}{k}$ = # k-subsets of $[n]$

$\binom{n-1}{k-1}$ = # k-subsets S of $[n]$ with $n \in S$ ← why?

$\binom{n-1}{k}$ = # k-subsets S of $[n]$ with $n \notin S$ ← why? ✓ ☒

Boundary of P.Δ is all 1's since $\binom{n}{0} = \binom{n}{n} = 1$.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

?

Recurrence says you fill out P. Δ by adding entries like this

Some other basic properties of $\binom{n}{k}$'s you can notice from P.'s Δ:

Prop. (Symmetry) $\binom{n}{k} = \binom{n}{n-k}$

Pf: $\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}$ ✓ or $x-y$ symmetry in bin. thm. $\square$

or Bijective Proof??

Prop. (Unimodality)

For $k < n/2 - 1$, $\binom{n}{k} < \binom{n}{k+1}$.

(P. Δ gets bigger towards middle)

Pf: $\binom{n}{k} = \frac{n!}{k!(n-k)!} < \frac{n!}{k!(n-k)!} \underbrace{\frac{n-(k+1)}{k+1}}_{>1,}$ since $k < n/2+1$

$= \frac{n!}{(k+1)!(n-(k+1))!}$

$= \binom{n}{k+1}$. $\square$

"Cultural aside" (not in book...)

Thinking about the rough shape of nth row of P.'s Δ leads to important probability theory results.

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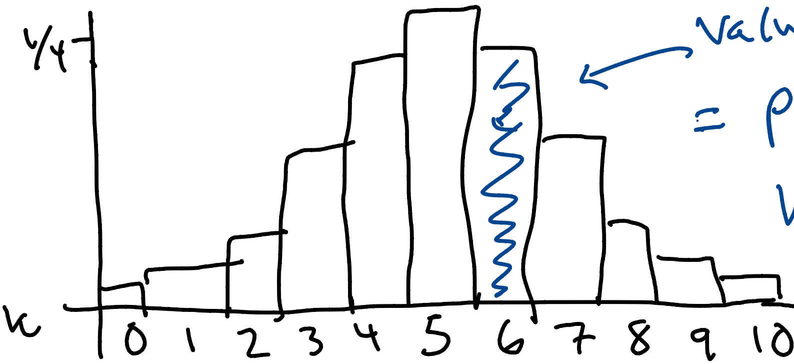
E.g., $n = 10$

histogram

depiction of 10th row of P. Δ

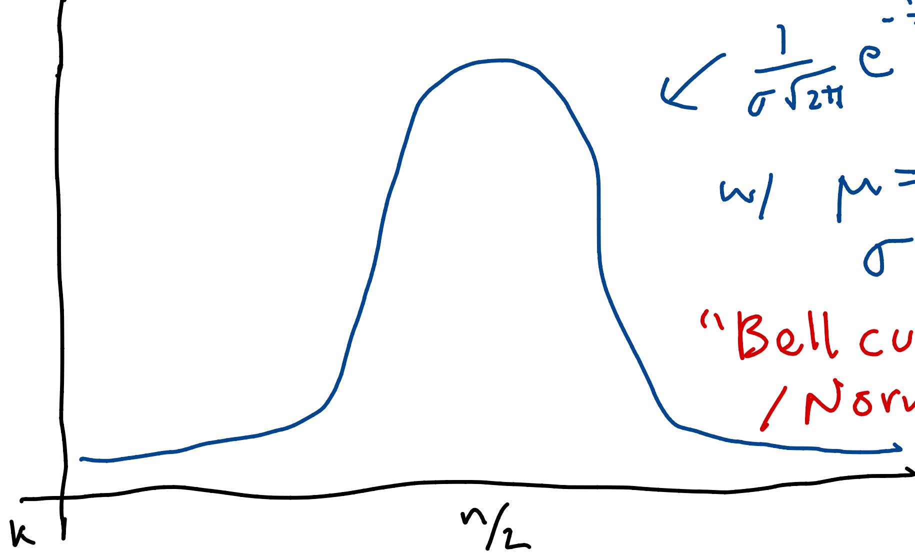
Divide bars by $2^n \sim 1000$:



value here = Prob. of getting $k = 6$ heads when flip $n = 10$ coins

Discuss this! Makes sense?

What does the histogram look like as $n \to \infty$?



$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

w/ $\mu = n/2$

$\sigma = \sqrt{n}/2$

"Bell curve" / Normal Distribution

—

Can be proved w/ Stirling's formula:

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

—

Tells us that when flip $n \gg 0$ coins:

- expect ratio of heads to be very close to $1/2$, — Law of Large #'s
- fluctuations of # heads from $n/2$ on order of $\sqrt{n}$. — Central Limit Theorem

The "Law of Large #'s" and "Central Limit Theorem" apply in a much broader context than flipping coins, and explain why science and social science work!

E.g., why

- If you average several measurements with error, you'll get close to true value.

- If you poll a reasonable # of people, can guess election result.

...

Now let's

take a break...

And when we come back

we'll do worksheet

on binomial coeff's

+ pascal's Triangle

in breakout groups.