Math 4990: Inclusion-Exclusion

10/13
Ch. 7

Reminders: - Should get HW#2 back soon if not already...

- Midterm #1 is due today!

We've already seen that to count some objects satisfying constraints, it's often easier to count objects violating the constraints, and then subtract.

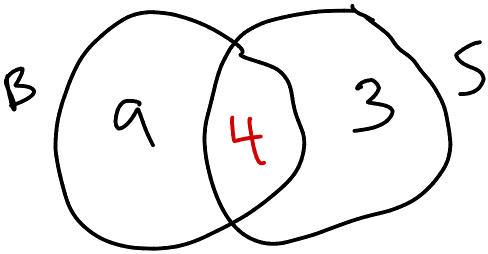
E.g., how many anagrams of BANANAS start with a letter other than A?

The principle of Inclusion-Exclusion is a formalization + extension of this idea, which allows for multiple constraints.

<u>Problem 1</u>: 13 students in a class play basketball, 7 play soccer, 4 play both. How many play either?

<u>Answer</u>: $13 + 7 - 4$ ← basketball ← soccer ← because of "double-counting"

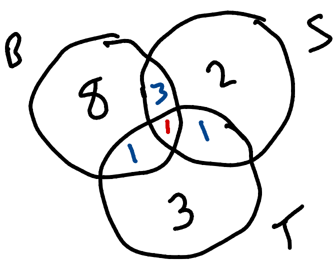
Easiest to visualize with a Venn diagram:



$$9 + 3 + 4 = (9+4) + (3+4) - 4 = 16$$

<u>Problem 2</u>: 13 play basketball, 7 play soccer, 6 play tennis, 4 play B+S, 2 play B+T, 2 play S+T, 1 plays all 3. How many play any of these sports?

<u>Answer</u>: $13 + 7 + 6 - 4 - 2 - 2 + 1$ ← dbl-counting ← we subtracted too much!



$$8 + 2 + 3 + 3 + 1 + 1 + 1 = (8+3+1+1) + (2+3+1+1) + (3+1+1+1) - (3+1) - (1+1) - (1+1) + 1 = 19$$

Same idea of subtracting (excluding) when we've double-counted, but then adding back (including) when we've subtracted too much will work for any # of sports.

But it's helpful to state general result formally...

Thm (Principle of Inclusion-Exclusion)

For sets $A_1, A_2, \ldots, A_k$,

$$\# A_1 \cup A_2 \cup \cdots \cup A_k = \sum_{\phi \neq I \subseteq [k]} (-1)^{\# I - 1} \# \bigcap_{i \in I} A_i.$$

E.g. $\# A_1 \cup A_2 \cup A_3 = \# A_1 + \# A_2 + \# A_3$
$- \# A_1 \cap A_2 - \# A_1 \cap A_3 - \# A_2 \cap A_3$
$+ \# A_1 \cap A_2 \cap A_3$

Pf: See book. But it boils down to

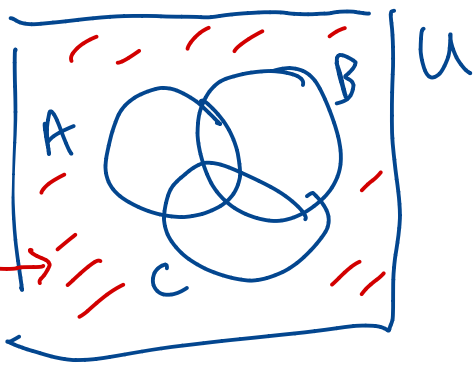
(*) $\sum_{i=0}^{k} (-1)^k \binom{n}{k} = 0$, which we proved. $\boxed{\checkmark}$

NOTE: Commonly $A_1, A_2, \dots, A_k$ represent "bad" properties you're trying to avoid.
So the following version most used:

Cor Let $A_1, A_2, \dots, A_k \subseteq U$, then ← 'universe'

$$\# U - (A_1 \cup \cdots \cup A_k) = \# U + \sum_{\emptyset \neq I \subseteq [k]} (-1)^{\# I} \# \bigcap_{i \in I} A_i$$

E.g.,



$\# U - A \cup B \cup C = \# U - \# A - \# B - \# C$
$+ \# A \cap B + \# A \cap C + \# B \cap C$
$- \# A \cap B \cap C.$ ✓

Example of PIE: Derangements

A permutation $p \in S_n$ is called a **derangement** if it has no **fixed points**; $p(i) \neq i \ \forall \ i \in [n]$.

E.g. $n=2$ 21 | $n=3$ 231, 312

Q: How many derangements in S_n?

Thm # derangements in S_n =

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots \pm \frac{1}{n!}\right) = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

Pf: By PIE, # derangements in S_n =

$\#S_n - \#\{p \in S_n : p(1)=1\} - \#\{p \in S_n : p(2)=2\} - \cdots -$

$+ \#\{p \in S_n : p(1)=1, p(2)=2\} + \cdots \quad - \cdots \quad \cdots$

$$= \sum_{I \subseteq [n]} (-1)^{\#I} \cdot \#\{p \in S_n : p(i) = i \ \forall i \in I\}$$

← choose any perm. on $i \notin I$

$$= \sum_{I \subseteq [n]} (-1)^{\#I} (n - \#I)!$$

↙ group all I w/ $\#I = k$ together

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \cdot (n-k)! = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \quad \blacksquare$$

<u>Hat check problem</u>: (Imagine it's old times...)

100 people go a play and check their hats at the lobby. But the lobby guy forgets to tag the hats, so at end of night gives hats back to people at random.

<u>Question</u>: What's the probability that no one gets their own hat back?

Easy to see that this is asking:
what's Prob. a random $P \in S_n$ is a derangement?

So by our thm, hat check prob. =

$$\frac{n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}}{n!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$ ← what is this?

Recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ ← finite approx. w/ $x = -1$

$\Rightarrow$ hat check prob. $\sim \frac{1}{e} = 0.3678\ldots$

w/ $n=100$, approx. is very good. (even $n=10$)

2nd Example of PIE:

We can use PIE to get a pretty good formula for $S(n,k)$, 2nd kind Stirling #'s

(the ones counting set partitions)

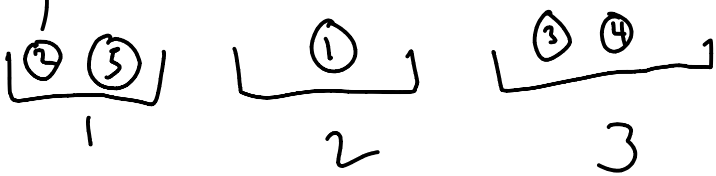
Thm $k! \cdot S(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$.

Pf: Recall that $k! \cdot S(n,k)$

$=$ # surjections $f : [n] \to [k]$

$=$ # ways of placing n distinct balls into k distinct boxes, where every box has at least one ball

e.g.,



By PIE,

$=$ # ball placements where some boxes can be empty $-$ # placements w/ box 1 empty $-$ # w/ box 2 empty $- \cdots$

$+$ # w/ boxes 1+2 empty $+ \cdots - \cdots \quad - \cdots$

$$= \sum_{I \subseteq [k]} (-1)^{\#I} \cdot \# \text{ ways placing } n \text{ balls into } k \text{ boxes where boxes } i \in I \text{ are empty}$$

$$= \sum_{I \subseteq [k]} (-1)^{\#I} \cdot (k - \#I)^n$$

each ball can go in any of boxes $i \notin I$

group I w/ $\#I = j$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n. \qquad \checkmark$$

E.g.

$$S(n,3) = \frac{1}{3!}\left(\binom{3}{0} 3^n - \binom{3}{1} 2^n + \binom{3}{2} 1^n - \binom{3}{3} 0^n \right)$$

0^n $= 0$ unless $n = 0$, then $= 1$

for $n \geq 1$, $S(n,3) = \dfrac{3^n - 3 \cdot 2^n + 3}{3!}$

$$S(2,3) = \frac{3^2 - 3 \cdot 2^2 + 3}{3!} = \frac{9 - 3 \cdot 4 + 3}{3!} = 0. \quad \checkmark$$

Now let's take a break!

And when we come back, do more PIE problems on the worksheet in breakout groups...