Math 4990: Generating functions

10/20
Ch. 8
(1st half)

Reminders: • HW #3 posted, due 10/27

• Should have Midterm 1 back soon, if not already

We've almost come to the end of our discussion of enumeration of basic combinatorial structures.
We'll end by explaining one of the most powerful techniques: generating functions.
G.f.'s can be mysterious... Let's start w/ an example.

Ex.: Fibonacci Numbers

Define a sequence of #'s F_n, $n \geq 0$ by

$$F_0 = 0,\ F_1 = 1,\ \overset{(*)}{F_n = F_{n-1} + F_{n-2}} \quad n \geq 2:$$

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

These Fibonacci #'s are the answer to many counting problems: e.g., you saw on HW#2 that

F_n = # set partitions of $[n+1]$ which have no singleton blocks.

The recurrence (*) makes it easy to compute the F_n, but we could still ask for a more explicit formula that tells us e.g. growth rate.

Def'n If a_n, $n \geq 0$ is a sequence of #'s, its (ordinary) generating function is

$$A(x) := \sum_{n \geq 0} a_n x^n.$$

You can either think of this as a formal expression (a power series) or a function of the parameter x (e.g., $x \in \mathbb{R}$ or $x \in \mathbb{C}$).

So let's form the g.f. of the Fibonacci #'s:

$$F(x) := \sum_{n \geq 0} F_n x^n.$$

The recurrence (*) let's us write:

$$F(x) = \sum_{n\geq 0} F_n x^n = 0 + x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n$$

$$= x + \sum_{n\geq 2} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n$$

$$= x + \sum_{n\geq 1} F_n x^{n+1} + \sum_{n\geq 0} F_n x^{n+2}$$

$$= x + x F(x) + x^2 F(x)$$

So

$$-x^2 F(x) - x F(x) + F(x) = x$$

$$\Rightarrow \boxed{F(x) = \frac{x}{1-x-x^2}.}$$

OK... but so what? We found a closed expression for $F(x)$, but what does that tell us about the #'s F_n?

Actually, ... it tells us a lot!

Remember from basic calculus the geometric series $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$,

So $\frac{1}{1-cx} = 1 + cx + c^2x^2 + \cdots = \sum_{n \geq 0} c^n x^n$,

i.e., $\frac{1}{1-cx}$ is the generating fn. for powers of c.

But how is that useful for the Fib #'s

w/ $F(x) = \frac{x}{1-x-x^2}$?

Well first, let's observe

$$1 - x - x^2 = \Big(1 - \underbrace{\tfrac{1+\sqrt{5}}{2}}_{\phi}\, x\Big)\Big(1 - \underbrace{\tfrac{1-\sqrt{5}}{2}}_{\psi}\, x\Big)$$

How did I find this... ?

So $F(x) = \frac{x}{(1-\phi x)(1-\psi x)}$, but still

don't see the connection to geometric

series until we remember Partial fractions:

$$\frac{x}{(1-\phi x)(1-\psi x)} = \frac{A}{1-\phi x} + \frac{B}{1-\psi x}$$

$$\Rightarrow \quad x = (1-\psi x)A + (1-\phi x)B$$
$$= (A+B)1 + (-\psi A - \phi B)x$$

$$\Rightarrow A+B=0, \quad -\psi A - \phi B = 1$$

$$\ldots \Rightarrow \quad A = \frac{1}{\sqrt{5}}, \quad B = \frac{-1}{\sqrt{5}},$$

So finally,

$$\sum_{n\geq 0} F_n x^n = F(x) = \frac{1/\sqrt{5}}{1-\phi x} - \frac{1/\sqrt{5}}{1-\psi x}$$

$$= \frac{1}{\sqrt{5}} \sum_{n\geq 0} \phi^n x^n - \frac{1}{\sqrt{5}} \sum_{n\geq 0} \psi^n x^n$$

So extracting coefficient of x^n,

exact formula!

$$F_n = \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{1.618\ldots}^n - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{-0.618\ldots}^n$$

In particular, $F_n \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$ as $n \to \infty$

This same basic technique will work to give exact formula for any linear recurrence

e.g., $a_n = 2a_{n-1} + a_{n-2} - 3a_{n-3} + 5$

(will practice on worksheet...)

Hopefully starting to see power of generating functions!

What if we have multiple generating functions? How can they 'interact'?

$$A(x) = \sum_{n\geq 0} a_n x^n, \quad B(x) = \sum_{n\geq 0} b_n x^n$$

$$A(x) + B(x) = \sum_{n\geq 0} (a_n + b_n)\, x^n$$ ← meaning is pretty clear!

What about $A(x)B(x)$?

$$A(x)B(x) = (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= (a_0 b_0)1 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

$$= \sum_{n\geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n.$$ ← meaning less clear

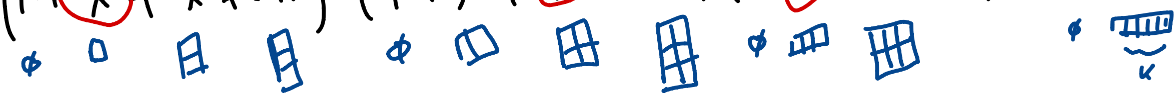
Let's see an example of using products of g.f.'s:

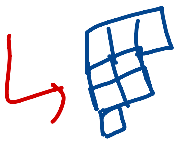
Let $P_{\leq k}(n) :=$ # Partitions of n w/ largest part $\leq k$

Prop. $$\sum_{n \geq 0} P_{\leq k}(n)\, x^n = \prod_{i=1}^{k} \frac{1}{1-x^i}$$

Pf. $$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \dots \cdot \frac{1}{1-x^k} =$$

$$(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\cdots(1+x^k+\dots)$$





Every choice of a term from each factor creates a unique partition, w/ exponent of n = size. ☐

Rmk: $P_{\leq k}(n)$ also = # partitions of n into $\leq k$ parts

Cor $$\sum_{n \geq 0} \underbrace{p(n)}_{\text{\# part. of } n} x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Pf: $k \to \infty$

Generating functions are also useful for proving partition identities.

Ex. Let $o_n = \#$ partitions of n into odd parts

$d_n = \#$ partitions on n into distinct parts

$$O(x) := \sum_{n\geq 0} o_n x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \cdots$$

why?

$$D(x) := \sum_{n\geq 0} d_n x^n = (1+x)(1+x^2)(1+x^3) \cdots$$

$\emptyset$ $\square$ $\emptyset$ $\square\!\square$ $\emptyset$ $\square\!\square\!\square$ etc.

Then,

$$D(x) \cdot \frac{(1-x)}{(1-x)} \cdot \frac{(1-x^2)}{(1-x^2)} \cdot \frac{(1-x^3)}{(1-x^3)} \cdot \frac{(1-x^4)}{(1-x^4)} \cdots$$

$$= (1+x)(1+x^2)(1+x^3)\cdots \frac{(1-x)}{(1-x)} \frac{(1-x^2)}{(1-x^2)} \frac{(1-x^3)}{(1-x^3)} \cdots$$

$$= \frac{(1\cancel{+x})(1\cancel{-x})}{(1-x)} \frac{(1\cancel{+x^2})(1\cancel{-x^2})}{(\cancel{1-x^2})} \frac{(1\cancel{+x^3})(1\cancel{-x^3})}{(1-x^3)} \frac{\cdots}{(\cancel{1-x^4})} \cdots$$

recall $(1-a)(1+a) = (1-a^2)$

$$\cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots = O(x). \Rightarrow \text{So } d_n = o_n \ \forall n.$$

(we saw this before)

What about if $A(x) = \sum_{n\geq 0} a_n x^n$ is a g.f. and we look at $\frac{1}{1-A(x)}$?

$$\frac{1}{1-A(x)} = 1 + A(x)^2 + A(x)^3 + A(x)^4 + \cdots .$$

Ex. Let $c_n :=$ # compositions of n.

Form g.f. $C(x) := \sum_{n\geq 0} c_n x^n$.

Then $C(x) = \dfrac{1}{1-\frac{x}{1-x}}$, since

$\frac{x}{1-x} = x + x^2 + x^3 + \cdots$ so

$$\frac{1}{1-\frac{x}{1-x}} = \frac{1}{1-(x+x^2+x^3+\cdots)} =$$

2+1

$\rightarrow 2$

$$\underset{\emptyset}{1} + (\underset{1}{x} + \underset{2}{x^2} + \underset{3}{x^3} + \cdots) + (\underset{1}{x} + \underset{2}{x^2} + \underset{3}{x^3} + \cdots)(\underset{1}{x} + \underset{2}{x^2} + \underset{3}{x^3} + \cdots)$$

1+2+1

$$+ (\underset{1}{x} + \underset{2}{x^2} + \cdots)(\underset{1}{x} + \underset{2}{x^2} + \cdots)(\underset{1}{x} + \underset{2}{x^2} + \cdots) + \cdots$$

Rewriting

$$C(x) = \frac{1}{1 - \frac{x}{1-x}} \cdot \frac{1-x}{1-x}$$

$$= \frac{1-x}{1-x \; -x} = \frac{1-x}{1-2x}$$

$$= \frac{1}{1-2x} - \frac{x}{1-2x} = \sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0} 2^n x^{n+1}$$

$$= \sum_{n\geq 0} 2^n x^n - \sum_{n\geq 1} 2^{n-1} x^n.$$

Extracting coeff. of x^n

$$\Rightarrow c_n = \begin{cases} 1 & \text{if } n=0 \\ \underbrace{2^n - 2^{n-1}}_{= 2^{n-1}} \checkmark & \text{if } n \geq 1 \end{cases}$$

We saw this before too!

Fun to see what other results we proved earlier can be proved with g.f.'s.

Now let's take a break...

And when we come back we can practice using generating functions in breakout groups with today's worksheet.