Math 4990: Catalan Numbers

10/27

Ch. 8 continued

Reminder:

- HW #3 due today (apologies again for the Q's on exponential generating functions...)

Last class we introduced generating functions.

There is so much more we can say about them... for instance if a_n, $n \geq 0$ is some sequence of numbers, we defined its ordinary generating function to be

$$A(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Its exponential generating function is

$$A(x) := \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

(Why "exponential"? Think $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.)

Exponential g.f.'s are useful if your a_n's grow fast, e.g. faster than c^n for any $c \in \mathbb{R}$, b/c then $\sum a_n x^n$ won't converge, but $\sum \frac{a_n}{n!} x^n$ might.

E.g. the Bell numbers $B(n) = \#$ set partitions $[n]$ has beautiful e.g.f. $\sum \frac{B(n)}{n!} x^n = e^{e^x - 1}$.

As a rule of thumb, e.g.f.'s are useful when:

- dealing with labelled structures,
- moving between connected structures and all structures (see the 'exponential formula').

You can read more about e.g.f.'s in the book... however, I decided that since today is our last day of enumeration we should do something more fun: Catalan numbers!

First let's go over something from last class's worksheet

Recall from calculus...

Thm (Taylor Series)

For a 'reasonable' function $f: \mathbb{R} \to \mathbb{R}$, have

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!},$$

where $f^{(k)} = k^{th}$ derivative of f.

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Let's take $f(x) = (1+x)^n$, where $n \in \mathbb{R}$ is any real number

e.g. $(1+x)^{-3} = \frac{1}{(1+x)^3}$, $(1+x)^{\frac{1}{2}} = \sqrt{1+x}$, $(1+x)^{\pi} = ???$

Remember from calculus that $f'(x) = n(1+x)^{n-1}$, and

$$f^{(k)}(x) = n \cdot (n-1) \cdots (n-(k-1)) (1+x)^{n-k},$$ so

Thm (Generalized binomial theorem)

For any $n \in \mathbb{R}$, $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$, where

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-(k-1))}{k!}.$$ ← generalized def. of binomial coeff.'s

NOTE: If $n \in \mathbb{N}$ is a nonnegative integer, then $\binom{n}{k} = 0$ when $k > n$, so we get as usual

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k. \checkmark$$

On the worksheet, it asked you to consider taking n to be a negative integer, e.g. $(1+x)^{-4} = \frac{1}{(1+x)^4}$.

Let's think about when n is a rational number:

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\cdots\left(-\frac{(2k-1)}{2}\right)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2k-1)}{2^k \quad k!} x^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{2^k k!} \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{2^k \quad k!} x^k$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} \left(-\frac{1}{4}\right)^k x^k$$

So... $(1-4x)^{-1/2} = \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{-1}{\cancel{4}}\right)^k (-\cancel{4}x)^k = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$,

the g.f. of central binomial coeff's!

$\binom{2k}{k} = 1, 2, 6, 20, 70, \ldots$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
⋮

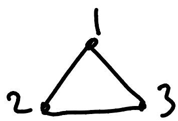
Rmk The g.f.'s we discussed earlier were all rational, i.e., ratios $\frac{P(x)}{Q(x)}$ of polynomials P, Q.

$(1-4x)^{-1/2} = \frac{1}{\sqrt{1-4x}}$ is not rational (it's algebraic).

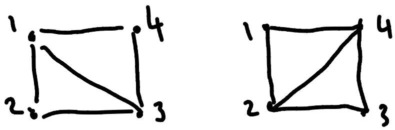
Now let's consider a new counting problem...

$C_n :=$ # triangulations of a $(n+2)$-gon.

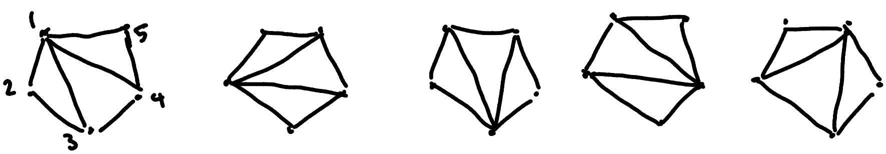
$C_1 = 1$



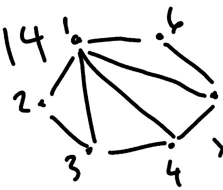
$C_2 = 2$



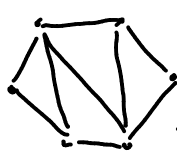
$C_3 = 5$



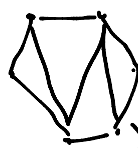
$C_4 = 14$



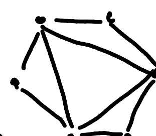
$\times 6$ rotations



$\times 3$



$\times 3$



$\times 2$

$C_5 = 42$ ... no way I'm drawing those!

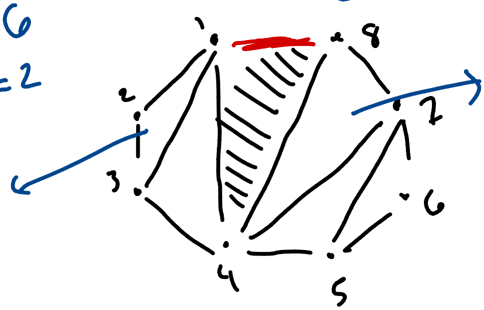
Also reasonable to define $C_0 = 1$

The C_n are called Catalan numbers.

Thm (Fundamental recurrence)

For $n \geq 1$, $C_n = \sum_{k=0}^{n-1} C_k C_{(n-1)-k}$.

Pf: By picture: e.g., $n=6$, $k=2$



"base" edge triangle splits any triangulation of an $(n+2)$-gon $\rightsquigarrow C_n$ into tri. of $(k+2)$-gon and $((n-1-k)+2)$-gon

$(k+2)$-gon $\to C_k$, $((n-1-k)+2)$-gon $\to C_{(n-1)-k}$

All choices of k and of the two smaller triangulations are possible, so

$$C_n = \sum_{k=0}^{n-1} C_k C_{(n-1)-k}, \text{ as claimed.}$$

$\boxed{/\!/}$

Okay, but what's the connection to g.f.'s?...

Remember that if $A(x)=\sum a_n x^n$ and $B(x)=\sum b_n x^n$

then $A(x)\,B(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n.$

So the fund. recurrence says something very nice about the Catalan number g.f.:

$$C(x)=\sum_{n=0}^{\infty} C_n x^n$$

namely,

$$C(x)C(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} C_k C_{n-k}\right) x^n$$

$$\text{(fund. rec.)} = \sum_{n=0}^{\infty} C_{n+1} x^n = \sum_{n=1}^{\infty} C_n x^{n-1}$$

$$= \frac{1}{x}\left(C(x)-1\right)$$

i.e., $x\,C(x)^2 - C(x) + 1 = 0$

$\Rightarrow C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$ by quad. form.

Remember,

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

$$\int (1-4x)^{-1/2} = \text{const.} + \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1}$$

$$\parallel$$

$$-\tfrac{1}{2}(1-4x)^{1/2} \quad \underset{\leadsto}{x=0} \quad \text{const.} = \tfrac{1}{2}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{2}\sqrt{1-4x} = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1}$$

$$\Rightarrow \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n$$

$$\parallel$$

$$\sum_{n=0}^{\infty} C_n x^n$$ (Since these coeff's are ≥ 0, shows we should take $-$ in $\pm$)

$$\Rightarrow \boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

e.g. $C_4 = \frac{1}{5}\binom{8}{4} = \frac{1}{5}\cdot 70 = 14$ ✓

$=$ # triang. of hexagon

So with generating functions we were able easily to find an explicit formula for Catalan numbers.

There are other ways to prove the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$

(can you find a bijective proof???)

but... this proof using g.f.'s is probably the "easiest."

Shows power of generating functions!

Now let's take a break...

And when we come back we can work in breakout groups on the worksheet, which shows many more counting problems where the answer is the Catalan #'s!