Ramsey theory,
UMTYMP Advanced Topics, Fall 2020

In lecture today we discussed Ramsey's theorem, which says that if we color the edges of a big enough complete graph red and blue, then it will contain a large monochromatic complete subgraph. But we could imagine instead of coloring the edges, we color the K_r-subgraphs, and again look for a large complete subgraph all of whose K_r-subgraphs are colored the same color. This is the content of the *generalized Ramsey's theorem*:

Generalized ("hypergraph") Ramsey's theorem. For any $r \geq 1$ and $k, \ell \geq r$, there exists a smallest integer $R_r(k, \ell)$ such that if we color all the K_r-subgraphs of K_N blue or red, where $N \geq R_r(k, \ell)$, then there exists either a K_k-subgraph of K_N all of whose K_r-subgraphs are colored blue, or a K_ℓ-subgraph of K_N all of whose K_r-subgraphs are colored red.

For example, the case $r = 2$ is about coloring edges, so $R_2(k, \ell) = R(k, \ell)$; and the case $r = 3$ is about coloring triangles. The proof is similar to the proof of Ramsey's theorem we gave, but with more notation.

Now we'll show an application of the case $r = 3$ to geometry. Let $ES(n)$ denote the smallest integer such that if we choose any $N \geq ES(n)$ points in the plane $\mathbb{R}^2$, with no three points collinear, then there are n of them which form the vertices of a **convex** n-gon. (We will show $ES(n)$ exists!)

1. Show $ES(4) = 5$ by drawing the possible point configurations.
2. Fix a configuration of points $1, 2, \ldots, N$ in the plane, no three of them collinear, and for each $1 \leq i < j < k \leq N$, color the triangle (i, j, k) blue if i, j, k appear in clockwise order in the triangle they form, and red if they appear in counterclockwise order. Show that if you take any four points $1 \leq i < j < k < \ell \leq N$ such that all four triangles (i, j, k), (i, j, ℓ), (i, k, ℓ), (j, k, ℓ) are the same color, then i, j, k, ℓ are the vertices of a **convex** quadrilateral.
3. Conclude that $ES(n) \leq R_3(n, n)$.

This geometry problem is called the **"happy ending problem"**, because it lead to the marriage of George Szekeres and Esther Klein.