Trees, UMTYMP Advanced Topics, Fall 2020

An *isomorphism* between graphs G and H is a one-to-one correspondence between the vertices of G and of H such that two vertices in G are joined by an edge if and only if their corresponding vertices in H are. If there is an isomorphism between G and H then we say that they are *isomorphic*.

1. If T is a tree and G is isomorphic to T, must G also be a tree?

2. *Cayley's formula* says that the number of **labelled** trees on n vertices is n^{n-2}. But many of those labelled trees will be isomorphic. How many non-isomorphic (**"unlabelled"**) trees are there on 4 vertices? On 5 vertices? If you're feeling really bored: on 6?

The *degree sequence* $(d_1, d_2, \ldots, d_n)$ of a graph G on n vertices is the sequence of degrees $\deg_G(v)$ for all vertices v of G, written in non-increasing order: $d_1 \geq d_2 \geq \cdots \geq d_n$.

3. Explain why isomorphic trees have the same degree sequences.

4. Find two non-isomorphic trees with the same degree sequences.

5. Explain why the degree sequence $(d_1, d_2, \ldots, d_n)$ of a tree T on n vertices is a non-increasing sequence of integers between 1 and $n-1$ such that $\sum_{i=1}^{n} d_i = 2(n-1)$.

6. Can you show that any non-increasing sequence $(d_1, d_2, \ldots, d_n)$ of integers between 1 and $n-1$ with $\sum_{i=1}^{n} d_i = 2(n-1)$ is the degree sequence of a tree on n vertices?

Recall that a *spanning tree* of a connected graph G is a subgraph that's a tree which contains all the vertices of G.

7. Find all the spanning trees of: