NOTES FROM THE AIM WORKSHOP ON DYNAMICAL ALGEBRAIC COMBINATORICS

These are notes from the "Dynamical algebraic combinatorics" workshop held March 23rd–27th, 2015 at the American Institute of Mathematics in San Jose, California. The organizers were James Propp, Tom Roby, Jessica Striker, and Nathan Williams. The reading list for the workshop included the papers [15] [25] [42] [54] [63]. The website for the workshop is `http://aimath.org/pastworkshops/dynalgcomb.html`. The problem session was moderated by Vic Reiner. These notes were recorded and typed up by Sam Hopkins: all mathematical insights are the speaker's; all mistakes are my own.

Notation used throughout:

- $\mathbb{R}$ is the set of real numbers, $\mathbb{Q}$ is the set of rationals, $\mathbb{Z}$ is the set of integers, and $\mathbb{N} := \{0, 1, 2, \ldots\}$;
- $\#X$ is the cardinality of the set X;
- $X \Delta Y$ denotes the symmetric difference of the sets X and Y;
- $\binom{X}{k}$ denotes the set of subsets of X of cardinality k and $2^X := \bigcup_{k \geq 0} \binom{X}{k}$;
- $\mathfrak{S}_X$ is the symmetric group on a set X;
- $[a, b] := \{a, a+1, \ldots, b-1, b\}$ (which is empty if $a > b$) and $[n] := [1, n]$;
- $\mathfrak{S}_n := \mathfrak{S}_{[n]}$ is the symmetric group on n letters;
- $\underline{n} := [n]$ viewed as a poset $1 < 2 < \cdots < n$ (i.e., a chain);
- a^b denotes the partition with b rows of length a;
- SYT means Standard Young Tableaux;
- SYT(λ) denotes the set of SYT of shape λ.

1. Monday morning lectures

1.1. Jessica Striker: Toggle group actions, applications, and abstractions.

Slides: `http://www.ndsu.edu/pubweb/~striker/prorowAIM.pdf`. Jessica explained the philosophy of toggle groups: we start with a set E, specify a set of subsets $\mathcal{L} \subseteq 2^E$, and then define toggles $\tau_e \colon \mathcal{L} \to \mathcal{L}$ for $e \in P$, which are involutions that have only a small, local effect. The toggle group is $\langle \tau_e \colon e \in P \rangle \subseteq \mathfrak{S}_{\mathcal{L}}$. The point is to study the action of interesting compositions of toggles on $\mathcal{L}$: orbit structure, cyclic sieving, homomesy, et cetera. (Often the first thing people prove about a toggle group is that it is always either the full symmetric group or the alternating group, but Jessica maintains that this is not the most interesting question to ask about toggles.)

First Jessica explained toggling of order ideals: our set is a poset P and set of subsets is $\mathcal{J}(P)$, the order ideals of P (otherwise known as downsets, i.e., subsets $I \subseteq P$ such that $x \in I$ and $y \leq x$ implies $y \in I$). The toggles $\tau_e \colon \mathcal{J}(P) \to \mathcal{J}(P)$ are

$$\tau_e(I) := \begin{cases} I \Delta \{e\} & \text{if } I \Delta \{e\} \in \mathcal{J}(P) \\ I & \text{otherwise} \end{cases}$$

for $e \in P$. She explained how Wieland's gyration operation [60] on fully-packed loops, or equivalently Alternating Sign Matrices (ASMs), can be seen as a composition of toggles in a certain tetrahedral poset P. Gyration exhibits resonance with pseudo-period $2n$, which means roughly that most orbits are of size a multiple of $2n$. An explanation for this resonance phenomenon is that gyration rotates the link pattern associated to a fully-packed loop. Another composition of toggles in this tetrahedral poset P, defined by Striker-Williams [54], is superpromotion, which exhibits resonance with pseudo-period $3n - 2$. Jessica put forward the problem of understanding this resonance in terms of cyclic rotation of some object associated to the order ideal. Jessica also explained that rowmotion of order ideals, and promotion of SYT of rectangular shape with two rows, can be seen as compositions of order ideal toggles. Specifically, rowmotion is a composition of toggles in P from top-to-bottom, and promotion is a composition of toggles from left-to-right (here "top-to-bottom" means in the reverse order of some linear extension of P; for making sense of "left-to-right" in a poset, see [54]).

Then Jessica gave some applications of toggles and the toggle group. One major application is producing equivariant bijections and demonstrating that various actions have equivalent orbit structure. Because rowmotion, promition, and gyration are conjugate to one another in the toggle group, these actions all have the same orbit structure. In this way one can show that rectangular SYT with two rows under promotion and "triangular" posets under rowmotion have the same orbit structure. Another application of the toggle group is establishing instances of the cyclic sieving phenomenon of Reiner-Stanton-White [43]. Jessica also explained an application to physics: in recent work [53] she has shown that the "togglability" statistic is homomesic with respect gyration or rowmotion for any poset, and this yields a new proof of one step in Cantini-Sportiello's proof [9] of the Razumov-Stronganov conjecture about the $O(1)$ loop model. The togglability statistic with respect to some element $e \in P$, which we denote $\mathfrak{T}_e \colon \mathcal{J}(P) \to \{-1, 0, 1\}$, is given by

$$\mathfrak{T}_e(I) := \begin{cases} 1 & \text{if } e \text{ can be toggled out of } I; \\ -1 & \text{if } e \text{ can be toggled into } I; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, Jessica ended with a discussion of generalized toggling for arbitrary subsets. The set-up is as in the first paragraph above: we have a ground set E, specify a set of subsets $\mathcal{L} \subseteq 2^E$, and define the toggles by

$$\tau_e(X) := \begin{cases} X \Delta \{e\} & \text{if } X \Delta \{e\} \in \mathcal{L}; \\ X & \text{otherwise.} \end{cases}$$

Unlike in the order ideal cases, the toggle group is not always isomorphic to $\mathfrak{S}_\mathcal{L}$ or the alternating group inside this symmetric group. However, what we still really should care about is compositions of toggles. Examples of places to look for interesting generalized toggles include other subsets associated to a poset such as chains, or subsets associated to other combinatorial objects such as independent sets of graphs or flats of matroids.

1.2. **Nathan Williams: Some combinatorial objects and actions in Coxeter groups.**

Nathan presented a 2×2 diagram, explained in much more detail in his thesis [62] and his 2014 FPSAC submission [63], of poset-theoretic and Coxeter-theoretic objects associated to any Coxeter group W:

$$
\begin{array}{ccc}
\mathcal{L} & \!\!\!\!\!-\!\!\!-\!\!\! & \mathcal{R} \\
\text{Poset} \Big| & & \Big| \text{Coxeter} \\
\mathcal{J} & \!\!\!\!\!-\!\!\!-\!\!\! & \mathcal{S}
\end{array}
$$

Here $\mathcal{L}$ stands for linear extensions of $\Phi^+(W)$, the positive root poset of W, $\mathcal{R}$ stands for reduced words of the longest word w_0, $\mathcal{J}$ stands for order ideals of $\Phi^+(W)$, and $\mathcal{S}$ stands for subword complexes of $\mathbf{c}$-sorting words. The point of this diagram is to draw an analogy between the top and the bottom rows. The major result concerning the top row is that when W is one of the "coincidental types" A_n, B_n, H_3 or $I_2(k)$ we have that $\#\mathcal{L}(W) = \#\mathcal{R}(W)$; this is a result due to several authors over a number of papers [49] [14] [28]. One way to prove this equality is via a bijection $\mathcal{L}(W) \to \mathcal{R}(W)$ induced by cyclic actions. Specifically promotion on $\mathcal{L}(W)$ is in equivariant bijection with *rotation* on $\mathcal{R}(W)$ (where the asterisks denote a technical correction when conjugation by w_0 does not act as the identity). For an example of how this promotion version of the Edelman-Greene bijection works, see [63, Example 4.3]. The idea of how the bijection is defined is that starting with some linear extension $L \in \mathcal{L}(W)$, you form a reduced word in $\mathcal{R}(W)$ by composing the simple reflections corresponding to minimal element of L, $\mathrm{Prom}(L)$, $\mathrm{Prom}^2(L)$, and so on in a full promotion orbit.

In order to describe the bottom row of this diagram, we need to fix a Coxeter word $\mathbf{c}$ (i.e., a product of all the simple reflections of W in some order). We then define $\mathbf{w_0}(\mathbf{c})$ to be the $\mathbf{c}$-sorting word; the definition of this word is a little complicated in general but in type A we can just think of it as $\mathbf{w_0}(\mathbf{c}) := s_1 \cdots s_n | s_1 \cdots s_{n-1} | \cdots | s_2 s_1 | s_1 |$. We then define $\mathcal{S}_{\mathbf{c}}(W, k)$ to be the set of subwords of $\mathbf{c}^k \mathbf{w_0}(\mathbf{c})$ that belong to $\mathcal{R}(W)$. Define $\mathcal{J}(W, k)$ to be the set of $\Phi^+(W)$-partitions of height k; i.e., $\mathcal{J}(W, k) := J(\Phi^+(W) \times \underline{k})$, the set of order ideals of the positive root poset times the chain $\underline{k}$. Again we have the miraculous theorem that for W a coincidental type, $\#\mathcal{J}(W, k) = \#\mathcal{S}_{\mathbf{c}}(W, k)$ for all $k \geq 1$ (see [55]).

Moreover, for *all* W, we have that $\#\mathcal{J}(W, 1) = \#\mathcal{S}_{\mathbf{c}}(W, 1)$, as proved by Armstrong-Stump-Thomas [1]. This theorem has its roots in map $\mathrm{Row} \colon \mathcal{J}(W, 1) \to \mathcal{J}(W, 1)$ due to Panyushev [39] and further explored, in the context of the cyclic sieving phenomenon, by Bessis-Reiner [4]. Even though this theorem has a uniform statement, it has resisted a uniform proof (the proof given in [1] uses parabolic induction). Nathan has proposed a uniform proof of this theorem akin to the bijection $\mathcal{L}(W) \to \mathcal{R}(W)$ induced by cyclic actions. Specifically, Nathan conjectures that there is a *Cambrian rotation* on $\mathcal{J}(W, 1)$ that induces a bijection from $\mathcal{J}(W, 1)$ to $\mathcal{S}_{\mathbf{c}}(W, 1)$. To define the Cambrian rotation, we define the sequence of positive roots $\mathbf{Camb_c} := \mathbf{Inv}(\mathbf{w_0}(\mathbf{c}))\mathbf{Inv}(\mathbf{w_0}(\mathbf{c}))^+$, where $\mathbf{Inv}(\mathbf{w_0}(\mathbf{c}))$ is the inversion sequence of $\mathbf{w_0}(\mathbf{c})$ and $\mathbf{Inv}(\mathbf{w_0}(\mathbf{c}))^+$ is this sequence with the simple roots removed. Then we define $\mathrm{Camb}_{\mathbf{c}} \colon \mathcal{J}(W, 1) \to \mathcal{J}(W, 1)$ to be the composition of toggles $\mathrm{Camb}_{\mathbf{c}} := \tau_{\alpha_n} \cdots \tau_{\alpha_1}$ where $\mathbf{Camb_c} = (\alpha_n, \ldots, \alpha_1)$. Here τ_α is just the order ideal toggle on $\mathcal{J}(\Phi^+(W))$ in the sense Jessica Striker explained above.

Again, the subwords in $\mathcal{S}_{\mathbf{c}}(W, 1)$ have a cyclic *rotation* action. The claim is that the Cambrian rotation and *rotation* actions are compatible (as was the case with promotion and *rotation*). For an example of how this Cambrian rotation bijection works, see [63, Example 4.6]. The idea of how the map is defined is that starting with some order ideal $I \in \mathcal{J}(W, 1)$, you form a subword in $\mathcal{S}_{\mathbf{c}}(W, 1)$ by keeping track of which simple roots belong to I, $\mathrm{Camb}_{\mathbf{c}}(I)$, $\mathrm{Camb}_{\mathbf{c}}^2(I)$, and so on in a full Cambrian rotation orbit. Of course the major outstanding problem is to construct the inverse of this map.

2. Monday afternoon problem session

2.1. Melody Chan: An interesting probability distribution on lattice paths.
Define $\Omega := \{\text{North/East lattice paths } (0,0) \to (a,b) \text{ in } \mathbb{Z}^2\}$. So $\#\Omega = \binom{a+b}{a}$. Define a probability distribution P on Ω as follows: let $T \in \mathrm{SYT}(a^b)$ be chosen uniformly at random and let $k \in [0, a \cdot b]$ be chosen uniformly at random; then the probability of a path $\mu \in \Omega$ with respect to P is the probability that T restricted to $[k]$ has shape μ. Let $\mathcal{U}$ be the uniform distribution on Ω. Define the following random variable:

$$C \colon \Omega \to \mathbb{N}$$
$$\mu \mapsto \text{\# of inside corners of } \mu + \text{\# of outside corners of } \mu$$
$$(= \text{\# of togglable elements in } \mu \text{ considered as an order ideal of } \underline{a} \times \underline{b})$$

The following is a (nontrivial) theorem of Chan, López, Pflueger, Teixidor [11]:

$$\mathbb{E}C \text{ on } (\Omega, P) = \frac{2ab}{a+b} = \mathbb{E}C \text{ on } (\Omega, \mathcal{U}).$$

Question: Can we understand this random variable C on (Ω, P) better, e.g., compute its higher moments? Is there some way of understanding this result via homomesy? Can we compute this same expectation but with lattice paths inside some shape λ other than $\lambda = a^b$?

2.2. Sam Hopkins: Homomesy in perfect matchings and oscillating tableaux.
Consider the set $\mathcal{M}_n := \{\text{perfect matchings of } [2n]\}$ and the three statistics $\mathcal{M}_n \to \mathbb{N}$

$$a(M) := \text{\# of alignments of } M;$$
$$c(M) := \text{\# of crossings of } M;$$
$$n(M) := \text{\# of nestings of } M.$$

'Clearly' $a(\cdot)$, $c(\cdot)$ and $n(\cdot)$ have the same mean. But they are not symmetrically distributed. However, there does exist an involution $\sigma \colon \mathcal{M}_n \to \mathcal{M}_n$ due to De Médicis-Viennot [12] such that $c(\sigma(M)) = n(M)$ and $n(\sigma(M)) = c(M)$.
Question: Is there a map $\tau \colon \mathcal{M}_n \to \mathcal{M}_n$ of order three that is homomesic with respect to $al(\cdot)$? Can one achieve $\langle \tau, \sigma \rangle \simeq \mathfrak{S}_3$? If so this would be a non-cyclic (indeed non-abelian) instance of homomesy.

Set $\mathcal{OT}(\lambda, k) := \{\text{oscillating tableaux of length } k \text{ and shape } \lambda\}$. For an oscillating tableau $T = (\emptyset = \lambda_0, \lambda_1, \ldots, \lambda_k = \lambda) \in \mathcal{OT}(\lambda, k)$ define $wt(T) := \sum_i |\lambda^i|$. There exists a bijection $RS \colon \mathcal{M}_n \to \mathcal{OT}(\emptyset, 2n)$ sending $a(\cdot)$ to a linear function of $wt(\cdot)$.

Question: Is there an order three map $\mathcal{OT}(\lambda, |\lambda| + 2n) \to \mathcal{OT}(\lambda, |\lambda| + 2n)$ homomesic for $wt(\cdot)$? From Hopkins-Zhang [30] we know that three times the average of $wt(\cdot)$ over all $T \in \mathcal{OT}(\lambda, |\lambda| + 2n)$ is integral for any λ and n.

2.3. Travis Scrimshaw: Symmetry of area and bounce via toggles.

Let $\mathcal{D}_n := \{$Dyck paths of semilength $n\}$. Define a partial order $<$ on $\mathcal{D}_n$ where $D_1 \lessdot D_2$ if $\text{area}(D_1) = \text{area}(D_2) - 1$ and $\text{bounce}(D_1) = \text{bounce}(D_2) + 1$. See [26, §3] for a definition of these statistics on Dyck paths.

Problem: Find a symmetric chain decomposition of $(\mathcal{D}_n, <)$. This would yield a combinatorial proof that area and bounce are symmetrically distributed [26, Open Problem 3.11]. Anne Schilling and Travis Scrimshaw have a proposed map that gives this decomposition and which is "toggle-like." It works up to $n = 12$.

2.4. Travis Scrimshaw: Orbit structure of the zeta map.

Consider Haglund's zeta map $\zeta \colon \mathcal{D}_n \to \mathcal{D}_n$ which sends the pair of statistics (dinv, area) to (area, bounce) [26, Theorem 3.15].

Question: Empirically ζ appears to have many orbits of size two. Can we explain why this is? More generally, can we classify the orbits of ζ?

2.5. Hugh Thomas: Rowmotion peridiocity from representation theory.

Grinberg-Roby's proof [24] of periodicity for birational rowmotion on $P = \underline{p} \times \underline{q}$ was inspired by Volkov's proof [59] of the Zamolodchikov conjecture for Y-systems of type $A_p \times A_q$. B. Keller [33] has a different, more general proof of the Zamolodchikov conjecture using the representation theory of finite dimensional algebras.

Question: Can we use Keller's proof as inspiration for a different proof of Grinberg-Roby's result? Can it prove periodicity for some other posets P? Does Keller's proof suggest a tropical rowmotion on dimension vectors of representations?

2.6. Oliver Pechenik: Promotion of increasing tableaux.

Set $\text{Inc}(\lambda, n) := \{$increasing tableaux of shape λ and entries in $[n]\}$. (For background on these tableaux, see [40].) There are obvious analogs $BK_i \colon \text{Inc}(\lambda, n) \to \text{Inc}(\lambda, n)$ of Bender-Knuth involutions turning i's into $(i + 1)$'s and vice-versa where possible:

$$
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 4 \\
\cline{1-2}
3 \\
\cline{1-1}
\end{array}
\quad \xmapsto{BK_3} \quad
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 4 \\
\cline{1-2}
4 \\
\cline{1-1}
\end{array}
$$

So we can define an analog of promotion $\text{Pr} := BK_{n-1} \circ \cdots \circ BK_2 \circ BK_1$.

Question: What is the order of Pr on $\text{Inc}(\lambda, n)$? For $\lambda = k^2$ the order is n. For $\lambda = k^3$, it seems like the order may still be n. For $\lambda = k^l$ for $l \geq 4$ it is no longer true that the order is n, but it seems that promotion resonates with pseudo-period n.

2.7. Ben Young: The Novelli-Pak-Stoyanovskii bijection via toggles.

Let $f^\lambda := \#\text{SYT}(\lambda)$. The well-known Hook Length Formula says $f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}$. NPS [37] prove this formula via a bijection

$$
\{\text{labelings of } \lambda \text{ by } [n]\} \xrightarrow{NPS} \left\{ (P, Q) \colon P \in \text{SYT}(\lambda); Q \in \prod_{u \in \lambda} [-\text{leg}(u), \text{arm}(u)] \right\}
$$

in which P is produced via a series of jeu-de-taquin slides.

Question: Is there a toggling proof of this bijection that replaces the jeu-de-taquin with toggles (like one does for promotion)?

2.8. Zachary Hamaker: Piecewise-linear rowmotion as a linear combination of combinatorial rowmotion.

Let P be a finite poset and $\mathcal{J}(P)$ its set of order ideals. Recall the rowmotion map $\mathrm{Row} \colon \mathcal{J}(P) \to \mathcal{J}(P)$ as well as its piecewise-linear lifting $\mathrm{Row}^{PL} \colon \mathbb{R}^P \to \mathbb{R}^P$. Given a vector $v \in \mathbb{R}^P$ one can (in many ways) express $v = c_1 X_{I_1} + \cdots + c_t X_{I_t}$ with $I_j \in \mathcal{J}(P)$ and $X_I := \sum_{i \in I} x_i$.

Question: Let $P := \underline{p} \times \underline{q}$. Can we always find $c_1, \ldots, c_t$ such that for all $k \geq 0$ we have $(\mathrm{Row}^{PL})^k(v) = c_1 X_{\mathrm{Row}^k(I_1)} + \cdots + c_t X_{\mathrm{Row}^k(I_t)}$. Unfortunately, even when v lies in the order polytope $\mathcal{O}(P) := \{x \in \mathbb{R}^P \colon 0 \leq x_i \leq 1, x_i \leq x_j \text{ if } i \leq j\}$ of P, one sometimes needs to choose some c_i's negative.

2.9. Nicolas Thiey: Random-to-random linear extension shuffles.

This question is based on joint work with Arvind Ayyer and Anne Schilling. For P a naturally-labeled finite poset on $[n]$, define

$$\mathcal{L}(P) := \{\text{linear extensions } w = (w_1, \ldots, w_n) \text{ of } P \text{ viewed as elements of } \mathfrak{S}_n\}.$$

For $\{i, j\} \in \binom{[n]}{2}$ with $i < j$ define the random-to-random shuffle $\tau_{ij} \colon \mathcal{L}(P) \to \mathcal{L}(P)$ by $\tau_{ij} := s_j s_{j+1} \cdots s_{n-1} \cdots s_{i+1} s_i$ where

$$s_i(w) := \begin{cases} (w_1, \ldots, w_{i+1}, w_i, \ldots, w_n) & \text{if this lies in } \mathcal{L}(P) \\ \mathrm{w} & \text{otherwise} \end{cases}.$$
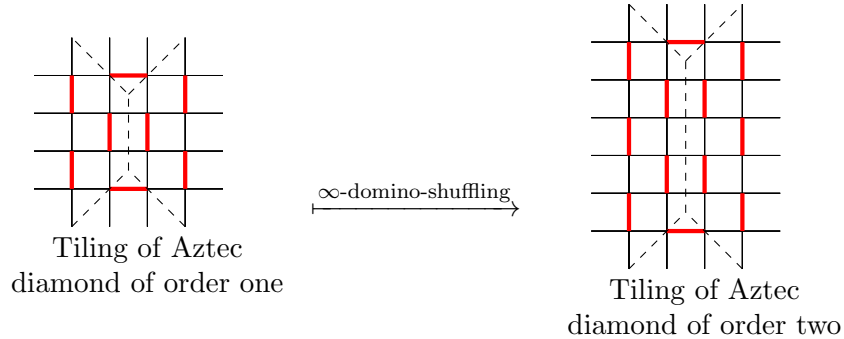
Define a Markov chain on $\mathcal{L}(P)$ that applies $\{\tau_{ij}\}$ with uniform distribution on $\binom{[n]}{2}$. Of course we have that the largest eigenvalue of the corresponding operator is $\lambda_{\mathrm{largest}} = 1$.

Conjecture: If P is disconnected, then $\lambda_{\text{2nd largest}} = \lambda(n) := (1 + \frac{1}{n})(1 - \frac{2}{n})$. If P is connected, then $\lambda_{\text{2nd largest}} < \lambda(n)$. For more details about this conjecture, see [2].

2.10. Gregg Musiker: Cluster mutations, domino-shuffling, and toggling.

Question: Can we understand g-vector mutations in cluster algebras in terms of domino-shuffling and/or toggling?

Gregg offered the following picture of the "∞-Aztec diamond":

Tiling of Aztec diamond of order one

$\xrightarrow{\infty\text{-domino-shuffling}}$

Tiling of Aztec diamond of order two

Note: it is unclear if we should be understanding cluster mutations as single toggles, a sequence of toggles, or birational toggles.

2.11. Luca Moci: Toggling in matroids.

Question: Given a matroid M on a ground set E, is there a family $\mathcal{L} \subseteq 2^E$ (such as independent sets, flats, etc.) and a way to do generalized toggling and rowmotion so as to achieve good behavior (predictable orbit sizes, homomesy, etc.)? Uniform matroids have no choice of ordering of E so are a natural place to start. James Propp also suggested starting with graphic matroids.

3. Tuesday Morning Lectures

3.1. James Propp: Dynamical algebraic combinatorics in the combinatorial and piecewise-linear realms.

Slides: `http://faculty.uml.edu/jpropp/dac.pdf`. Jim introduced the three realms in which we can investigate dynamical algebraic combinatorics: combinatorial, piecewise-linear and birational. In general proofs can go "downwards" (i.e., from birational) and ideas or inspiration can go "upwards" (from combinatorial). In particular, it appears that the only way to prove a positive result about piecewise-linear dynamics is to prove the analogous result for birational dynamics. Jim started by recalling the combinatorial realm: the set-up is that we have a finite set X, an invertible transformation $T \colon X \to X$, and (sometimes) a statistic $F \colon X \to \mathbb{R}$. The phenomena we can investigate are:

- **Periodicity**: $\forall x \in X, T^n x = x$;
- **Orbit-equivalence**: $\forall k \geq 0, \#\{x \colon T^k x = x\} = \#\{x' \colon (T')^k x' = x'\}$;
- **Cyclic sieving**: $\forall k \geq 0, \#\{x \colon T^k x = x\} = |p(\zeta^k)|$ where p is some polynomial (usually a generating function) and ζ is a primitive nth root of unity;
- **Invariance**: $\forall x \in X, F(Tx) = F(x)$;
- **Homomesy**: there exists $c \in \mathbb{R}$ such that for all orbits $\mathcal{O}$ of T in X, the average of $F(x)$ over $\mathcal{O}$ is c;
- **Reciprocity**: $\forall x \in X, F(T^k x) = -G(x)$ for appropriate G and k.

Jim offered a conjectural instance of homomesy. Set $E := \{(i, j) \colon i < j \in [n]\}$ and $\mathrm{NC}(n) := \{\text{noncrossing partitions of } [n]\}$. There is a map $\alpha \colon \mathrm{NC}(n) \to 2^E$ where we have $(i, j) \in \alpha(\Pi)$ iff $i < j$, i and j belong to the same part (say π_1) of Π, and there is no $k \in \pi_1$ with $i < k < j$. (This is just the "arc diagram" that corresponds to a noncrossing partition.) Clearly the map is injective. Let us do some generalized toggling in the sense of Striker above: take our ground set to be E and set of subsets $\mathcal{L}$ to be the image of α. Denote $\tau_{(i,j)} \colon \mathcal{L} \to \mathcal{L}$ by τ_{ij}. Note that

$$\kappa := \tau_{n-1,n} \circ \cdots \circ \tau_{2n} \circ \cdots \circ \tau_{23} \circ \tau_{1n} \circ \cdots \circ \tau_{13} \circ \tau_{12}$$

corresponds to Kreweras complementation. So κ has order $2n$. Let $F \colon \mathcal{L} \to \mathbb{N}$ be the statistic where $F(\Pi) = k$ if Π (thought of as a noncrossing partition) has k parts. Observe that $F(\Pi) + F(\kappa(\Pi)) = n + 1$ for any Π. So F is homomesic under κ with mean $\frac{n+1}{2}$. We can also define the same composition of toggles in a different order:

$$T := \tau_{1n} \circ \tau_{2n} \circ \tau_{1,n-1} \cdots \cdots \circ \tau_{n-2,n} \circ \cdots \circ \tau_{13} \circ \tau_{n-1,n} \circ \cdots \circ \tau_{23} \circ \tau_{12}.$$

Now T has a crazy and unpredictable orbit structure. But it remains true (conjecturally!) that $(\mathcal{L}, T, F)$ exhibits homomesy.

How do we get interesting invertible maps $T\colon X \to X$? One way, as in the previous example, is to compose involutions, i.e., toggles. Even though these toggles only make small changes, their compositions can be "big." Another way to obtain interesting invertible maps $T\colon X \to X$ is by composing various bijections. For instance, fix some poset P and consider the composition of bijections

$$\text{BS}\colon \mathcal{A}(P) \to \mathcal{J}(P) \to \mathcal{F}(P) \to \mathcal{A}(P).$$

(BS stands for Brouwer-Schrijver [7].) Here $\mathcal{A}(P)$ is the set of antichains of P, $\mathcal{J}(P)$ is the set of order ideals (or down-sets) of P, and $\mathcal{F}(P)$ is the set of filters (or up-sets) of P. The first map $\mathcal{A}(P) \to \mathcal{J}(P)$ is given by taking the downward closure of the antichain; the second map $\mathcal{J}(P) \to \mathcal{F}(P)$ is given by complementation; and the third map is given $\mathcal{F}(P) \to \mathcal{A}(P)$ by taking minimal elements. The overall map is rowmotion of antichains. (Note that we know from Cameron-Fon-Der-Flaass [8] that rowmotion is also a composition of toggles, so often these perspectives for generating T coincide.) We can also consider the composition of the same bijections in a different order:

$$\text{CF}\colon \mathcal{J}(P) \to \mathcal{F}(P) \to \mathcal{A}(P) \to \mathcal{J}(P).$$

(CF stands for Cameron-Fon-Der-Flaass [8].) Here we have changed the set X from $\mathcal{A}(P)$ to $\mathcal{J}(P)$, but the dynamics of the maps BS and CF are of course the same.

How do we find interesting homomesies given X a finite set and $T\colon X \to X$ invertible? Often we start with a vector space V of some functions $F\colon X \to \mathbb{R}$ that we think of as a "feature space." The point is that sometimes by taking linear combinations of statistics that are not homomesic, we arrive at one that is. For any example of this, let $X := \binom{[n]}{k}$ and $T\colon X \to X$ be cyclic rotation (i.e., $T(S) = \{s+1 \mod n \colon s \in S\}$). Then neither $\min\colon X \to \mathbb{R}$ nor $\max\colon X \to \mathbb{R}$ is homomesic with respect to the action of T, but $\min + \max$ is. Another important observation is that what qualifies as a "feature" can depend on how we view the space X and the map T: the BS and CF maps are formally equivalent, but the number of elements of an antichain in a given fiber (diagonal line in the Hasse diagram) is homomesic with respect to BS, while the number of elements of an order ideal in a given column (or "file") is homomesic with respect to CF. Along these lines, Jim proposed an investigation of how the homomesies proven by Propp-Roby [42] and by Bloom-Pechenik-Saracino [5] are related.

Then Jim proceeded to explain the piecewise-linear realm of dynamical algebraic combinatorics, at least as it applies to rowmotion of order ideals. Define a *framed P-partition with ceiling* n to be a weakly order-reversing map from P to $\{0, 1, \ldots, n\}$; denote the set of such maps by $\text{FPP}(P, n)$. Note that the indicator function of an order ideal is a framed P-partition with ceiling 1. For $x \in P$, define the P-partition toggle $\tau_x\colon \text{FPP}(P, n) \to \text{FPP}(P, n)$ by $\tau_x(f) := f'$ where
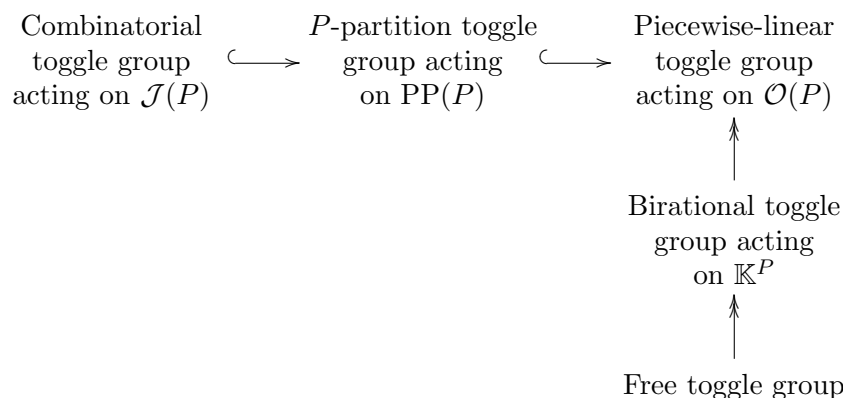
$$f'(y) := \begin{cases} f(y) & \text{if } y \neq x; \\ a + b - f(x) & \text{if } y = x \end{cases}$$

where $a := \max\{f(y)\colon y \gtrdot x\}$ and $b := \min\{f(y)\colon y \lessdot x\}$. We can then define rowmotion and promotion on P-partitions by composing these toggles in the appropriate order. (Note that this promotion is equivalent to promotion on semistandard Young tableaux

in the sense of Schützenberger; see `http://jamespropp.org/gtt-promotion.txt` for a detailed explanation of the connection.) Also, if we define $\mathrm{PP}(P)$ to be the disjoint union of $\mathrm{FPP}(P, n)$ for all $n \geq 0$, we can extend the toggles to $\mathrm{PP}(P)$ by having τ_x act separately as an involution each $\mathrm{FPP}(P, n)$; but note that the group generated by all the toggles $\tau_x \colon \mathrm{PP}(P) \to \mathrm{PP}(P)$ is now in general an *infinite* group.

P-partitions quickly yield a geometric picture for rowmotion. Let us scale down the entries in $\mathrm{FPP}(P, n)$ to $\{0, 1/n, 2/n, \ldots, 1\}$. Then $\mathrm{PP}(P)$ is the set of all order-reversing maps from P to $[0, 1] \cap \mathbb{Q}$; in other words, it is the set of rational points inside the order polytope $\mathcal{O}(P)$ of P. (Actually, Stanley [50] defined the order polytope $\mathcal{O}(P)$ to be the set of order-*preserving* maps from P to $[0, 1]$ but the distinction is technical.) The piecewise-linear toggles τ_x make sense on all of $\mathcal{O}(P)$; in fact, they become *fiber-flipping*: that is, to apply τ_x we rigidly reverse all the line segments in $\mathcal{O}(P)$ in the direction of the coordinate $x \in P$. Note that the vertices of the order polytope correspond precisely to $\mathcal{J}(P)$ and toggling at these vertices is the same as the toggling for order ideals in the sense of Striker-Williams [54]. So we can consider rowmotion (or promotion, or gyration, etc.) on $\mathcal{O}(P)$. As in the P-partition case, rowmotion may have infinite order: a nontrivial fact is that when it has infinite order on $\mathcal{O}(P)$ then there exists at least one infinite orbit; but the union of the infinite orbits need not be dense. The piecewise-linear realm lets us get closer to a precise notion of "resonance." Specifically, let us define the *spectrum* of a piecewise-linear map $\varphi \colon \mathcal{O}(P) \to \mathcal{O}(P)$ to be those $n \in \mathbb{N}$ for which the points of order n form a set of positive measure. For instance, when P is the tetrahedral poset associated to ASMs of order 4 and φ is piecewise-linear gyration, then $\mathrm{Spec}(\varphi) = \{8, 24, \ldots\}$. When $\mathrm{Spec}(\varphi) \neq \emptyset$ it makes sense to say that φ *resonates* with pseudo-period $\gcd(\mathrm{Spec}(\varphi))$. This is an instance of the piecewise-linear realm explaining phenomenon observed at the combinational realm.

Jim ended with a description of how all these toggle groups relate to one another:

$$
\begin{array}{ccccc}
\text{Combinatorial} & & P\text{-partition toggle} & & \text{Piecewise-linear} \\
\text{toggle group} & \hookrightarrow & \text{group acting} & \hookrightarrow & \text{toggle group} \\
\text{acting on } \mathcal{J}(P) & & \text{on } \mathrm{PP}(P) & & \text{acting on } \mathcal{O}(P)
\end{array}
$$

$$\uparrow$$

$$
\begin{array}{c}
\text{Birational toggle} \\
\text{group acting} \\
\text{on } \mathbb{K}^P
\end{array}
$$

$$\uparrow$$

$$\text{Free toggle group}$$

The birational toggle group will be explained in Tom Roby's talk below. The free toggle group is the group generated by τ_x for $x \in P$, with $\tau_x^2 = 1$, and $\tau_x \tau_y = \tau_y \tau_x$ for x and y that are not adjacent in the Hasse diagram, but subject to no further relations.

3.2. **Tom Roby: Birational rowmotion.**

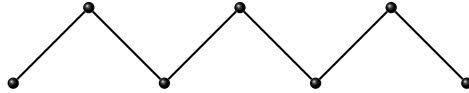Slides: `http://www.math.uconn.edu/~troby/homomesy2015aim.pdf`. First Tom reviewed classical rowmotion in the Stiker-Williams [54] sense and its appearances in

various guises [7] [19] [8] [39] [1]. Then Tom recalled the piecewise-linear version of toggling and rowmotion [15] that James Propp just defined in his last talk. What is birational rowmotion? First recall that *tropicalization* is the process of transforming a rational subtraction-free expression by replacing all instances of + with max and all instances of · with +; in other words, it is the process of moving from the $(+, \cdot)$ semiring to the $(\max, +)$ semiring. *Detropicalization* is the reverse procedure: if we have some expression involving max and +, we replace all instances of max with + and all instances of + with ·. (Note that $\min(z_i) = -\max(-z_i)$, so expressions involving min are also allowed.) But the definition of piecewise-linear toggle given above used only max and +! Thus we can define birational rowmotion by formally detropicalizing the piecewise-linear toggles.

Let P be a poset and $\mathbb{K}^P$ the set of maps from P to some field $\mathbb{K}$. It is necessary for technical reasons to work with the poset $\widehat{P}$ where we add a minimal element 0 and maximal element 1. Then we define the rational toggle $T_v \colon \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ for $v \in P$ by formally detropicalizing the piecewise-linear definition; that is

$$(T_v f)(w) := \begin{cases} f(w) & \text{if } w \neq v; \\ \dfrac{1}{f(v)} \dfrac{\sum_{u \lessdot v} f(u)}{\sum_{u \gtrdot v} \frac{1}{f(u)}} & \text{if } w = v. \end{cases}$$

And we define birtaional rowmotion $R := T_{v_1} \circ \cdots \circ T_{v_n} \colon \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ where $(v_1, \ldots, v_n)$ is some linear extension of P. (Note that we never toggle at 0 or 1). It is straightforward to show that $\operatorname{ord}(r) \mid \operatorname{ord}(R)$, where r is classical combinatorial rowmotion. Do we always have equality? No! For instance the following poset

has infinite order under birational rowmotion. Nevertheless for many "nice" posets we do have that the order of birational rowmotion is the same as the order of classical rowmotion.

Tom went on to explain the results obtained by Grinberg-Roby [24] about posets that have finite order under birational rowmotion. All posets are assumed to be ranked. For these investigations it is convenient to work with a "rank-homogenized" version of rowmotion: let $\overline{\mathbb{K}^{\widehat{P}}}$ be the quotient of $\mathbb{K}^{\widehat{P}}$ by the equivalence relation where $f \sim g$ if we can (separately) rescale each rank of f to obtain g. Then Grinberg-Roby show that

$$\begin{array}{ccc} \mathbb{K}^{\widehat{P}} & \overset{R}{\dashrightarrow} & \mathbb{K}^{\widehat{P}} \\ \pi \Big\downarrow & & \Big\downarrow \pi \\ \overline{\mathbb{K}^{\widehat{P}}} & \underset{\overline{R}}{\dashrightarrow} & \overline{\mathbb{K}^{\widehat{P}}} \end{array}$$

commutes (where $\overline{R}$ is the rank-homogenized version of birational rowmotion), and hence $\operatorname{ord}(\overline{R}) \mid \operatorname{ord}(R)$. One simple class of posets that have finite order under birational rowmotion is graded forests: for these we have $\operatorname{ord}(R) = \operatorname{ord}(r) \mid \operatorname{lcm}(1, \ldots, n+1)$ where n is the rank of the forest. The proof is essentially inductive. More complicated posets require more advanced techniques. One of the main theorems of [24] is that
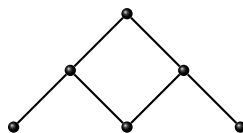
for $P = \underline{p} \times \underline{q}$, we have $\text{ord}(R) = p+q$. Moreover, Grinberg-Roby establish a reciprocity result: for $f \in \mathbb{K}^{\widehat{P}}$ and $i \in [p], k \in [q]$ we have

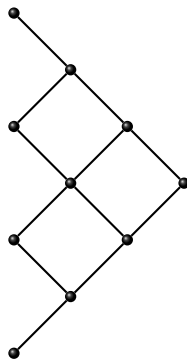$$f((p+1-i, q+1-k)) = \frac{f(0)f(1)}{(R^{i+k-1}f)(i,k)}.$$

The inspiration for their proof was Volkov's proof of the Zamolodchikov's $A_p \times A_q$ periodicity conjecture. The idea is to reparameterize $f \in \mathbb{K}^{\widehat{P}}$ by $p \times (p+q)$ matrices, taking quotients of maximal minors by "cycling" through the columns, and then use the 3-term Plücker relations. Specifically, for a matrix $A \in \mathbb{K}^{p \times (p+q)}$ and for each $j \in \mathbb{Z}$, we define $\text{Grasp}_j(A) \in \mathbb{K}^{\widehat{P}}$ (which stands for "Grassmannian parametrization") to be a certain homogenous quotient of maximal minors of A. The key steps of the proof are then to show

    (1) $\text{Grasp}_j(A) = \text{Grasp}_{p+q+j}(A)$ for all j and A;
    (2) $R(\text{Grasp}_j(A)) = \text{Grasp}_{j-1}(A)$ for all j and A;
    (3) for almost every $f \in \mathbb{K}^{\widehat{P}}$ satisfying $f(0) = f(1) = 1$, we have an A such that $\text{Grasp}_0(A) = f$;
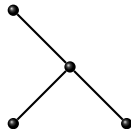   (4) in proving that $\text{ord}(R) = p+q$, we can assume that $f(0) = f(1) = 1$.

Tom went on to address some other posets where finite order either holds or is conjectured to hold. For the following "half-square" (i.e., triangle)

we have $\text{ord}(R) = 2p$ (in the picture $p = 4$). For this other half-square

we also have $\text{ord}(R) = 2p$ (again, in the picture $p = 4$). For this quarter-square

it is conjectured that $\text{ord}(R) = p$ (again, in the picture $p = 4$). This has been proven for p odd; note that for p even the quarter-square is the type B positive root poset. Nathan Williams conjectures that certain trapezoidal posets should also have a finite

(and explicit) order under birational rowmotion. In general it seems that $\operatorname{ord}(R) < \infty$ when P is a positive root poset of coincidental type or a minuscule poset. However, the positive root poset corresponding to D_4 has infinite birational rowmotion order.

Tom mentioned that, in light of Iyudu-Shkarin's proof [32] of the Kontsevich periodicity conjecture for noncommutative birational transformations, it would be interesting to explore noncommutative birational toggles. Here it is important to multiply the terms in the right order to get anything reasonable. Finally, as to homomesy at the birational level: recall that the number of elements in each column is homomesic with respect to classical rowmotion on $\underline{p} \times \underline{q}$ [42]; for birtaional rowmotion, the geometric means of each column only depend on top and bottom elements. This follows from the reciprocity statement mentioned earlier.

4. Tuesday Afternoon Problem Selections

The following eight problems were nominated for groups:

(1) The noncrossing partition toggle homomesy problem (explained by James Propp above).
(2) Exploring the interesting distribution on lattice paths (explained by Melody Chan above).
(3) A conjecture of Nathan Williams: the expected number of braid moves in a reduced word in the commutation class of $s_1 \cdots s_n s_1 \cdots s_{n-1} \cdots s_1 s_2 s_1$ is one. Here s_i are the standard generators (adjacent transpositions) of the symmetric group $\mathfrak{S}_{n+1}$. See [44] for some motivation for this problem.
(4) The $3n - 2$ superpromotion resonance problem (explained by Jessica Striker above).
(5) Extensions of the Grinberg-Roby proof of periodicity for birational rowmotion: using ideas from Keller (as suggested above by Hugh Thomas) to establish periodicity in more Coxeter-theoretic posets, extending recent work of Rush [45] to capture homomesy at the birational level, et cetera.
(6) The $\mathcal{J}(\Phi^+(W)) \to \mathcal{S}_{\mathbf{c}}(W)$ bijection (explained by Nathan Williams above).
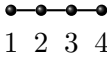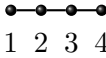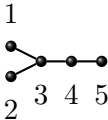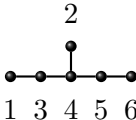(7) Resonance in promotion of increasing tableaux (explained by Oliver Pechenik above).
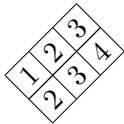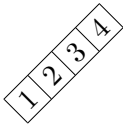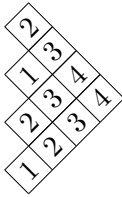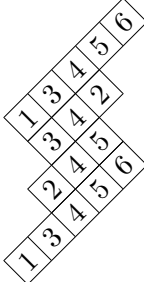
All but (4) were chosen as groups.

5. Wednesday Morning Lectures

5.1. **David Rush: Rowmotion in minuscule posets.**

David introduced minuscule posets. They depend upon a choice of complex simple Lie algebra $\mathfrak{g}$ (or equivalently, an irreducible root system, or still equivalently, a Dynkin diagram) as well as a choice of fundamental weight of $\mathfrak{g}$ to be the minuscule weight λ. This minuscule weight is the highest weight of a certain irreducible representation, called a minuscule representation, of $\mathfrak{g}$. A minuscule poset also comes with a natural "heap" labeling (the poset together with this labeling is called a heap). Heaps were introduced by Viennot [58] and extensively developed in the context of combinatorial Coxeter theory by Stembridge [52]. For background on heaps, see [46, §13] or the recent book [23]. David gave some examples of heaps for various choices of root system and

minuscule weight λ, where we picture the heaps as posets on boxes in such a way that a box is covered by the boxes to its north-east and north-west and the heap label is written inside the box:

Root system	A_4	A_4	D_5	E_6
Dynkin diagram				
Minuscule weight	$\lambda = \omega_2$	$\lambda = \omega_1$	$\lambda = \omega_1$	$\lambda = \omega_1$
Heap				

Note that the $\underline{k} \times \underline{(n-k)}$ poset corresponds to the representation $\Lambda^{n-k}(\mathbb{C}^n)$ of $\mathfrak{sl}_n(\mathbb{C})$. Also note that in this case, the heap labels correspond to columns; in general, the heap labeling may be seen as a generalization of the notion of columns to non-rectangular posets.

David explained the main thing he will investigate about a minuscule poset P is rowmotion $\Phi \colon \mathcal{J}(P) \to \mathcal{J}(P)$ on the order ideals of P. The only fact about rowmotion that David will really need is that the minimal elements of P/I are the maximal elements of $\Phi(I)$. As an immediate consequence, we see that for any $p \in P$, p can be toggled into I if and only if p can be toggled out of $\Phi(I)$. The main phenomenon related to rowmotion we are interested in are the cyclic sieving phenomenon and the homomesy phenomenon. Cyclic sieving for rowmotion in minuscule posets was established by Rush-Shi [46], but homomesy was only very recently established by Rush [45]. The key to both of these results is Stembridge's bijection [52] between order ideals of minuscule posets and fully commutative Weyl group elements explains the anatomy of the heap labeling.
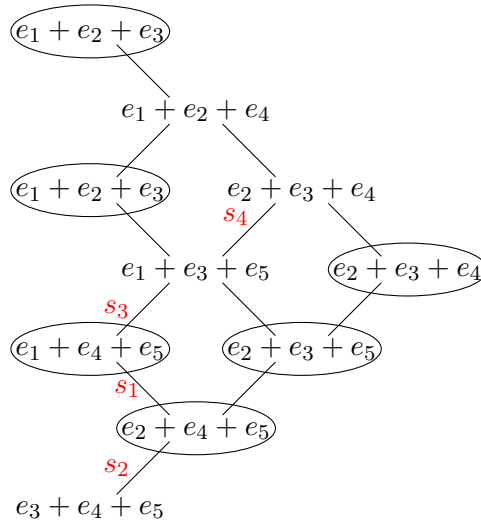
David then explained specifically his homomesy results: for a minuscule poset, the average number of elements labeled by the simple root α_i in a Φ-orbit is $\frac{2(\lambda, \omega_i)}{(\alpha_i, \alpha_i)}$ (here ω_i is a fundamental weight, to be explained in a moment.) Moreover, in the simply-laced cases we have that the average cardinality of an order ideal in a Φ-orbit is $\frac{(\lambda, \rho)}{(\alpha, \alpha)}$ (here ρ is the half-sum of fundamental weights), and the average cardinality of an antichain in a Φ-orbit is $\frac{2(\lambda, \lambda)}{(\alpha, \alpha)}$.

David then reviewed the basic set-up of simple finite dimensional Lie algebras and their representations. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{h}$ a choice of Cartan subalgebra. We have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where R is the set of roots
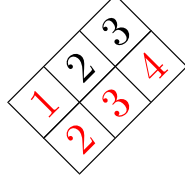
of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root space corresponding to α. Here R is a finite subset of $\mathfrak{h}^*$ that spans $\mathfrak{h}^*$. The killing form $(,)$ on $\mathfrak{g}$ induces an inner produce on $\mathfrak{h}^*_\mathbb{R}$ and thus we can speak of the inner product of roots. Let $\pi = \{\alpha_1, \ldots, \alpha_t\}$ be a choice of simple roots. The simple coroots are $\pi^\vee := \{\alpha_i^\vee\}$ where $\alpha_i^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. The lattice of weights of $\mathfrak{g}$ is then $\Lambda := \{\lambda \in \mathfrak{h}^*_\mathbb{R} : (\lambda, \alpha_i^\vee) \in \mathbb{Z}\}$. A weight $\lambda \in \Lambda$ is called *dominant* if $(\lambda, \alpha^\vee)$ is nonnegative for all $\alpha^\vee \in \pi^\vee$. These dominant weights are nonnegative linear combinations of the fundamental weights $\{\omega_1, \ldots, \omega_t\}$ where $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$. Recall that if λ is dominant, there exists a unique up to isomorphism simple representation of $\mathfrak{g}$ with highest weight λ denoted V^λ. Recall that $V^\lambda := \bigoplus_{\mu \in \Lambda_\lambda} V^\lambda_\mu$, where $\Lambda_\lambda \subseteq \Lambda$ is the set of weights of V^λ and V^λ_μ is the weight space corresponding to μ. Recall that the Weyl group W of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h}^*_\mathbb{R})$ generated by the simple reflections s_α for $\alpha \in R$, where for $\lambda \in \mathfrak{h}^*_\mathbb{R}$ we have $s_\alpha(\lambda) := \lambda - (\lambda, \alpha^\vee)\alpha$.

Finally we are ready to define the minuscule property of representations. A minuscule weight is one for which V^λ has a transitive Weyl group action on its weight spaces. In other words, λ is minuscule if $\Lambda_\lambda \subseteq W\lambda$. For any V^λ, we define a poset structure on Λ_λ whereby $\mu \lessdot \nu$ if the difference $\mu - \nu$ is a simple root. A key fact is that for λ minuscule we have $(\mu, \alpha_j^\vee) \in \{-1, 0, 1\}$ for all $\mu \in \Lambda_\lambda$ (this fact is apparently proven in Bourbaki [6]). Thus if $\mu - \nu = \alpha_i$ then $(\mu, \alpha_i^\vee) = 1$ and $(\nu, \alpha_i^\vee) = -1$. So we see that s_i exchanges μ and ν. Thus for λ minuscule, the edges of the Hasse diagram of Λ_λ are naturally labeled by simple reflections. As for minuscule posets: it turns out that Λ_λ is a distributive lattice (see [41]). Thus the join irreducibles of Λ_λ form a poset P_λ whose order ideals are in bijection with elements of Λ_λ (see [51, §3.5]). A *minuscule poset* is the poset of join irreducibles P_λ of Λ_λ for λ minuscule.

David then gave an example to make all of this comprehensible. Let us take $\mathfrak{g} = \mathfrak{sl}_5$ and $\mathfrak{h} = \{\text{traceless diagonal matrices}\}$. So $\mathfrak{h}^* = \mathbb{C}^5/(1, 1, \ldots, 1)$. The roots are the images in $\mathfrak{h}^*$ of $\{\pm(e_i - e_j) : 1 \le i < j \le 5\}$ under the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ induced by the Killing form. The simple roots are $\pi = \{e_2 - e_1, e_3 - e_2, \ldots, e_5 - e_4\}$. Choose as minuscule weight $\lambda := \omega_2$. Then $V^\lambda = \Lambda^3(\mathbb{C}^5)$. The poset Λ_λ is

Note that the minuscule weight $\lambda = e_3 + e_4 + e_5$ is the minimum of Λ_λ. The join irreducibles are circled. The corresponding minuscule poset P_λ is

Observe that a linear extension of P corresponds to a maximal chain of Λ_λ. Moreover, a linear extension of an order ideal $I \in \mathcal{J}(P_\lambda)$ corresponds to a saturated chain culminating at the element of Λ_λ that I represents. For instances, if we take I to be the downward closure of the elements labeled 1 and 4 in P_λ (the elements of I are in red) and take the linear extension $(2, 1, 3, 4)$ of P_λ restricted to I, then this corresponds to the chain whose edges are labeled s_2, s_1, s_3, s_4 in Λ_λ above (where we write s_i for s_{α_i}). Thus we see that the heap labeling of Stembrige [52] realizes the bijection between linear extensions of order ideals and saturated chains culminating at a particular element.

Rush-Shi [46] establish another key fact about the heap labeling: its equivariance with respect to the toggle group action and Weyl group action. Specifically, define τ_i to be the composition of toggling at all the elements labeled by a α_i in the heap. Then the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{J}(P_\lambda) & \xrightarrow{\sim} & W\lambda \\
{\scriptstyle \tau_i}\downarrow & & \downarrow{\scriptstyle s_{\alpha_i}} \\
\mathcal{J}(P_\lambda) & \xrightarrow{\sim} & W\lambda
\end{array}
$$

From this fact, cyclic sieving follows easily.

David went on to explain how Stanley's "wishful thinking as a proof technique" lead to the proof of homomesy. How can we use root-theoretic data to check how many elements labeled by i are in an order ideal $I \in \mathcal{J}(P_\lambda)$? Let μ_I denote the element of $W\lambda$ corresponding to $I \in \mathcal{J}(P_\lambda)$. Note that $(\mu, \alpha_i^\vee)$ is 1 if we can toggle in something labeled by i, -1 if we can toggle something out labeled by i, and 0 otherwise. Since the element p toggles into I if and only if p toggles out of $\Phi(I)$, we know that $(\mu_I, \alpha_i^\vee) = 1$ if and only if $(\Phi(\mu_I), \alpha_i^\vee) = -1$. Thus $(\cdot, \alpha_i^\vee)$ is zero over an orbit. For proving homomesy, it would be great if $\sum_{i=1}^t (\mu, \alpha_i^\vee) c_i = (\mu, \omega_i)$ for an appropriate choice of constants c_i. This would be great because for $I \in \mathcal{J}(P_\lambda)$ we have $\mu_I = \lambda - a_1\alpha_1 \cdots - a_t\alpha_t$, so $(\mu_I, \omega_i) = (\lambda, \omega_i) - a_i(\alpha_i, \omega_i)$, and this coefficient a_i is precisely number of elements in I labeled by i. So how can we show these c_i exist? Consider the linear map $\theta \xmapsto{A} \sum_{i=1}^t (\theta, \alpha_i^\vee)\alpha_i^\vee$. Note A is invertible because it sends the basis $\{\omega_1, \ldots, \omega_t\}$ of $\mathfrak{h}^*$ to $\{\alpha_1^\vee, \ldots, \alpha_t^\vee\}$. So let $(\theta_1, \ldots, \theta_t)$ be sent by A to $(\omega_1, \ldots, \omega_t)$. We claim $\sum_{i=1}^t (\mu, \alpha_i^\vee)(\theta_i, \alpha_i^v) = (\mu, \omega_i)$. But this is clear because by definition of θ_i we have $\sum_{i=1}^t \alpha_i^\vee(\theta_i, \alpha_i^\vee) = (\mu, \omega_i)$. Thus

$$
\sum_{I \text{ in some } \Phi\text{-orbit}} \sum_{i=1}^t (\mu_I, \alpha_i^\vee)(\theta_i, \alpha_i^\vee) = \sum_{i=1}^t (\theta_i, \alpha_i^\vee) \sum_{I \text{ in some } \Phi\text{-orbit}} (\mu_I, \alpha_i^\vee)
$$

which we know is 0 by our earlier argument about the sum of $(\mu_I, \alpha_i^\vee)$ along a rowmotion orbit. Putting all this together, we see that the average number of elements whose heap label is i in an Φ-orbit of $\mathcal{J}(P_\lambda)$ is $\frac{2(\lambda,\omega_i)}{(\alpha_i,\alpha_i)}$. To get the antichain homomesy is a little more complicated and requires computing a quadratic expression of inner products with coroots.

5.2. **Soichi Okada: On the existence of generalized parking spaces.**

Okada presented joint work with his student Y. Ito [31]. Let PF_n be the set of parking functions of length n, i.e., $\mathrm{PF}_n := \{(a_1, \ldots, a_n) \colon 1 \leq a_i \leq n, \#\{j \colon a_j \leq i\} \geq i\}$. Recall that $\mathfrak{S}_n \curvearrowright \mathrm{PF}_n$ by permuting entries and this action is isomorphic to the action $\mathfrak{S}_n \curvearrowright (\mathbb{Z}/n\mathbb{Z})/(1^n)$. (Note that $\#\mathrm{PF}_n = (n+1)^{n-1}$.) The permutation character of this action is given by $w \mapsto (n+1)^{\ell(\rho_w)-1}$ where ρ_w is the cycle type of w. Similarly, if $\gcd(k,n) = 1$ then $\mathfrak{S}_n \curvearrowright (\mathbb{Z}/k\mathbb{Z})/(1^n)$ has the permutation character $w \mapsto k^{\ell(\rho_w)-1}$. (If $\gcd(k,n) \neq 1$, this is not necessarily the character.)

Now let W be a (finite) complex reflection group and V its reflection representation. Let k be a positive integer. Define the following class functions

$$\varphi_k(w) = k^{\dim(V^W)} \text{where } V^W := \{v \in V \colon w \cdot v = v\}$$

$$\widetilde{\varphi_k}(w) = \frac{\det(1 - q^k w)}{\det(1 - qw)}.$$

Observe that $\lim_{q \to 1} \widetilde{\varphi_k}(w) = \varphi_k(w)$. The main questions Okada is interested in are:

- when is φ_k is the character of some representation of W;
- when is $\widetilde{\varphi_k}$ the graded character of some graded representation of W?

If k is a multiple of the Coxeter number of W, then φ_k is the character of some representation; see also the work of Geck-Michel [21] when W is a Weyl group. Define the q-numbers

$$\mathrm{Cat}_k(W,q) := \prod_{i=1}^r \frac{[k+d_i-1]}{[d_i]};$$

$$\mathrm{Cat}_k^*(W,q) := \prod_{i=1}^r \frac{[k+d_i^*-1]}{[d_i]} q^N;$$

wher $d_1, \ldots, d_r$ are the degrees of W, $d_1^*, \ldots, d_r^*$ are the codegrees, N is the number of reflections in W, and $[r] := \frac{1-q^r}{1-q}$. Ito-Okada have following classification: for W irreducible, the following are equivalent

(i) $\widetilde{\varphi_k}$ is the graded character of some representation of W;
(ii) $\mathrm{Cat}_k^*(W,q)$ is a polynomial in q;
(iii) some explicit condition on k is satisfied, i.e., for $W = \mathfrak{S}_n$ we have $\gcd(k,n) = 1$, and for $W = G(m,p,n)$ with $n \geq 3$ we have $k \cong 1 \mod m$.

Moreover, if W is not a dihedral group, then these conditions are equivalent to

(iv) φ_k is the character of some representation of W;
(v) φ_k is a permutation character.

Note that even though the statement is (mostly) uniform, the proof is not. Some implications are easy (e.g., $(i) \Rightarrow (ii), (i) \Rightarrow (iv), (v) \Rightarrow (iv)$), but the rest require case-by-case checking. The exceptional cases require computer assistance. If $W = \mathfrak{S}_n$, we can use properties of Schur functions to prove this classification.

Okada went on to explain the specifics of the proof for $W = \mathfrak{S}_n$. Note that here we have

$$\widetilde{\varphi_k} = \sum_{\lambda \vdash n} \frac{s_\lambda(1, q, \ldots, q^{k-1})}{[k]} \chi^\lambda$$

$$\varphi_k = \sum_{\lambda \vdash n} \frac{s_\lambda(\overbrace{1, \ldots, 1}^{k})}{[k]} \chi^\lambda$$

where s_λ is the Schur function and χ^λ the irreducible character of $\mathfrak{S}_n$. For which k, n are these coefficients in $\mathbb{N}[q]$ or $\mathbb{N}$? The answer is that

(i) $\gcd_{\mathbb{Z}}\{s_\lambda(\overbrace{1, \ldots, 1}^{k}): \lambda \vdash n\} = \frac{k}{\gcd(n,k)}$;

(ii) $\gcd_{\mathbb{Q}[q]}\{s_\lambda(1, q, \ldots, q^{k-1}): \lambda \vdash n\} = \frac{[k]}{[\gcd(n,k)]}$.

These gcd computations easily imply the classification theorem mentioned above. Note that

$$\frac{[\gcd(n,k)]}{[k]} s_\lambda(1, q, \ldots, q^{k-1}) \in \mathbb{N}[q].$$

Haiman [27] has shown that if $\gcd(n,k) = 1$ this polynomial has unimodal coefficients. Okada conjectures the following: set $\sum_{i \geq 0} a_i q^i = \frac{[\gcd(n,k)]}{[k]} s_\lambda(1, q, \ldots, q^{k-1})$; then the sequences $(a_0, a_2, \ldots)$ and $(a_1, a_3, \ldots)$ are both unimodal. In the case $k \mid n$ this is classical. It is known for $\gcd(n,k) = 1$ via the representation theory of rational Cherednik algebras (see [3]).

Okada finished with the following problem: if λ has one row, then $s_\lambda(1, \ldots, q^{k-1})$ is the q-binomial coefficient given by q-counting Dyck paths. Okada asked if we can find a combinatorial interpretation of

$$\frac{[\gcd(a,b)]}{[a+b]} \begin{bmatrix} a+b \\ a \end{bmatrix}$$

via q-counting. When $\gcd(a, b) = 1$ we can use rational Dyck paths. Even the $q = 1$ case of this problem is open when $\gcd(a, b) \neq 1$, however.

6. Wednesday Afternoon Problem Selection

There was a call for new problems to be suggested, and one was:

6.1. Darij Grinberg: Two conjectures of Schützenberger.

These conjectures are apparently due to Schützenberger. They appear in [13]. Let us say the SYT T_1 and T_2 of shape λ *differ by a cycle* if and only if their reading words $w(P_1)$, $w(P_2)$ (with respect to some fixed convention) have $w^{-1}(T_1)w(T_1)$ a cycle, not necessarily of consecutive values (e.g. $(2547)(1)(6)(3)$).

Conjecture 1: If T_1, T_2 differ only in the positions of i and $i+1$, then $\mathrm{evac}(T_1)$ and $\mathrm{evac}(T_2)$ differ by a cycle of even length.

Conjecture 2: If T_1, T_2 are, in addition, of rectangular shape, then for each k, $\mathrm{Prom}^k(T_1)$ and $\mathrm{Prom}^k(T_2)$ differ by a cycle of even length.

After the call for new problems and a summary of group progress from the day before, the following problems were selected as groups:

(1) The noncrossing partition toggle homomesy problem.

> The group reported that they were investigating products of toggles of all the arcs in any arbitrary order. They call the resulting maps "Coxeter toggles" in the sense of Coxeter element. Miriam Farber has computational evidence that any Coxeter toggle exhibits homomesy with the number of parts statistic. The group wanted to continue attacking this conjecture as well as understand when two Coxeter toggles have the same orbit structure.

(2) The interesting distribution on lattice paths.

> The group reported that they computed the expected number of corners for a lattice path inside λ according to Melody Chan's interesting distribution when λ is a hook. They want to continue exploring the expectation for other shapes.They also want to look for explanations of these expectations via homomesy, perhaps using Suter's cyclic action on a portion of Young's lattice [56].

(3) The expected number of braid moves in a reduced word in a certain commutation class.

> The group reported that they reformulated this question and discovered it was really a question about shifted staircase tableaux. Vic Reiner asked whether the variance of the number of braid moves for reduced words in this commutation class might be computed, and if we should expect this random variable to behave like a Poisson random variable.

(4) The $3n-2$ superpromotion resonance problem.

> This group subsumed the increasing tableaux group from the day before. The increasing tableaux group reported that they found an equivariant bijection between increasing tableaux and plane partitions, and this bijection partially explains some resonance phenomenon of promotion in plane partitions (i.e., order ideals in $\underline{a} \times \underline{b} \times \underline{c}$ exhibit resonance with pseudo-period $a+b+c-1$ under promotion). Jessica Striker suggested that this bijection might help in attacking the $3n-2$ problem as well, so that was the direction the group was headed in.

(5) Connections between birational rowmotion and cluster algebras/Y-systems.

> This group was one of two groups that splintered off from the birational rowmotion group from last time. They reported that they have understood the $A_p \times A_q$ Y-system as essentially the same as the "homogenous" version of birational rowmotion. Their goal is to understand cluster mutations as birational toggles in other settings.

(6) The $\mathcal{J}(\Phi^+(W)) \to \mathcal{S}_{\mathbf{c}}(W)$ bijection.

> The group reported that they have done some computations by hand and are close to getting code written that can test many cases of the conjectured bijection. Their goal is to try to factor Cambrian rotation in some other way than as the composition of toggles described above.

(7) The conjectures of Schützenberger (explained by Darij Grinberg above).

(8) Extensions of birational rowmotion to the minuscule setting.

> This was the other group that splintered off from the birational rowmotion group. Their goal was to extend Grinberg-Roby's work to the minuscule poset setting explored by Rush-Shi [46] and Rush [45].

7. Thursday Morning Lecture

7.1. **Gregg Musiker: Cluster algebras.**

Gregg first reviewed the periodicity conjecture and its history. Let Δ, Δ' be Dynkin diagrams on vertex sets I, I' and let C, C' be the corresponding Cartan matrices. Define $A = (a_{i,j}) := 2\mathrm{Id}_{\#I} - C$ and $A' = (a'_{i',j'}) := 2\mathrm{Id}_{\#I'} - C'$. Here are some examples of these matrices:

$$\Delta = A_5 \Rightarrow C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

$$\Delta = B_5 \Rightarrow A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

$$\Delta = C_5 \Rightarrow A = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let h, h' denote the Coxeter numbers of Δ, Δ' (recall that Δ gives rise to a Weyl group W; the Coxeter number h is the order in W of a Coxeter element, which is a product of all the simple reflections in any order). The $\Delta \times \Delta'$ Y-system is then the collection $\{Y_{i,i',t} : (i,i') \in I \times I', t \in \mathbb{Z}\}$ satisfying the relations

$$Y_{i,i',t+1} Y_{i,i',t-1} = \frac{\prod_{j \in I}(1 + Y_{j,i',t})^{a_{i,j}}}{\prod_{j' \in I'}(1 + Y_{i,j',t}^{-1})^{a'_{i',j'}}}.$$

(We can think of the $Y_{i,i',t}$ as positive real numbers or rational functions.) The periodicity conjecture, proved by Keller [33], is that $Y_{i,i',t+2(h+h')} = Y_{i,i',t}$. Gregg summarized the history of this conjecture, following the introduction of [33]: in 1991, motivated by the thermodynamic Bethe ansatz, Zamolodchikov [64] conjectured that $\Delta \times A_1$ case for Δ simply connected. The $A_n \times A_1$ case was proven with explicit solutions by Frenkel-Szenes [20] and independently by Gliozzi-Takeo [22] using volumes of three-folds and

triangulations. The $\Delta \times A_1$ case (without the assumption that Δ be simply-laced) was proven by Fomin-Zelevinsky [17] using cluster algebras. The $A_n \times A_m$ case was proven by Volkov [59] using explicit solutions by cross-ratios/determinants; and later by Henriques [29] using the bounded octahedron recurrence and Szenes [57] using flat connections on a graph. Finally, Keller [33] proved the general case using ideas from cluster algebras and categorification.

Gregg then took a moment to explain why we care about the Y-system at this workshop. The key observation of Max Glick, Darij Grinberg, and possibly others, is that when $P = p \times q$, $f(i, i')$ are rational functions on the vertices of the Hasse diagram of P, and $R \colon \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ is birational rowmotion on P, then we have

$$Y_{i,i',i+i'-2k} = \frac{R^k f(i, i'+1)}{R^k f(i+1, i')}$$

where the $Y_{i,i',t}$ belong to the $A_{p-1} \times A_{q-1}$ Y-system. In other words, not only is the Grinberg-Roby [24] proof of finite order of birational rowmotion on P inspired by the proof of Volkov [59] for periodicity of the $A_n \times A_m$ Y-system, but the two dynamical systems are in fact formally related.

Gregg then explained Fomin-Zelevinksy's cluster algeba approach [17] to Y-systems using quivers and Y-system mutation, as seen through the lens of Keller [33]. Let Q be a quiver on $[n]$, i.e. Q is directed multigraph with no 2-cycles with vertex set $[n]$. For $k \in [n]$, we define the quiver mutation $\mu_k(Q) := Q'$ where Q' is obtained from Q by:

(1) for each $i \to k \to j$ in Q, add $i \to j$;
(2) reverse all arrows $i \to k$ and $k \to j$;
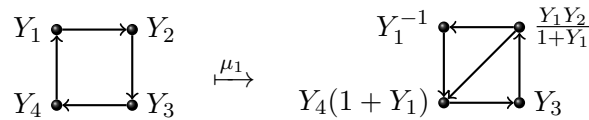(3) erase all 2-cycles.

An example of quiver mutation is the following:

Now let (Q, Y) be a pair where Q is a quiver and $Y = (Y_1, \ldots, Y_n)$ is a tuple of subtraction-free rational functions associated to the vertices of Q. For $k \in [n]$, we define the Y-system mutation μ_k by $\mu_k(Q, Y) := (Q', Y')$ where $Q' := \mu_k(Q)$ and

$$Y'_j := \begin{cases} Y_j^{-1} & \text{if } k = j \\ Y_j(1 + Y_k^{-1})^{-m} & \text{if } k \overset{m}{\Rightarrow} j \text{ (i.e., there are } m \text{ arrows from } k \text{ to } j) \\ Y_j(1 + Y_k)^m & \text{if } k \overset{m}{\Leftarrow} j \end{cases}$$
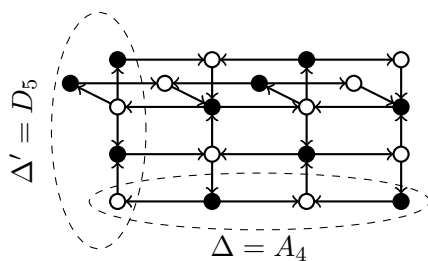
An example of Y-system mutation is the following:

Now we return to Dynkin diagrams as above. Pick bipartite orientation of Δ, Δ'; for example:
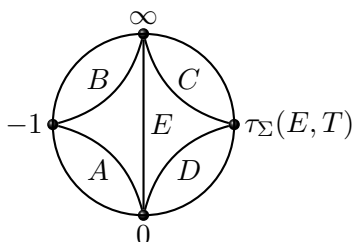

$$\Delta = A_4 \qquad \Delta' = D_5$$

Then form the square product quiver $\Delta \square \Delta'$ (a variant of the tensor product $\Delta \otimes \Delta'$); for example:
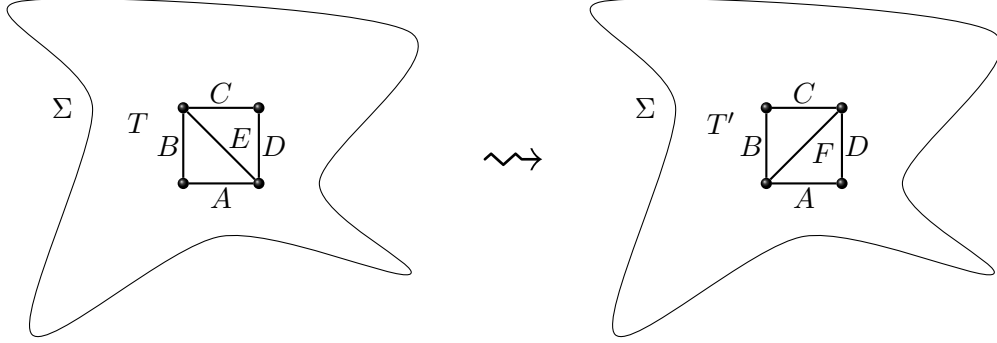

Note that in $\Delta \square \Delta'$ mutations at any black vertices commute, and mutations at any white vertices commute. Thus we can define τ_+ to be mutating at black vertices and τ_- to be mutating at all white vertices. The claim is then that $\tau_- \tau_+$ (which can be seen as a kind of "gyration" in the sense of Jessica Striker's talk above) corresponds to two time steps of the Y-system. Fomin-Zelevinsky [17] study the (Q, Y) mutations as well as the cluster mutations which keep track of more data. At a certain level, Y-system mutations can be seen as rotation of associahedra or other more general polytopes.

Gregg ended with a brief discussion of shear coordinates in hyperbolic geometry and topical shear coordinates (see [16, §12] for more details). The set-up here is that we have some surface Σ together with a triangulation T of Σ and we want to give coordinates $\tau_\Sigma(T, E)$ to the edges E of the triangulation. To do that we use *cross-ratios*: for a quadrilateral inside our triangulation with outer edges A, B, C, D and diagonal E we define $\tau_\Sigma(T, E)$ using the Poincaré disk model by mapping three of the vertices of the quadrilateral to $0, -1, \infty$ and seeing where the other vertex is mapped:
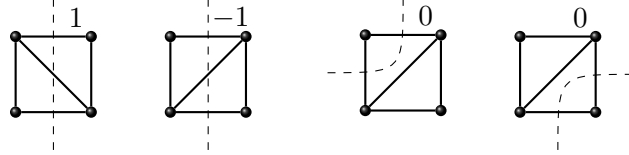

How do these shear coordinates change as the triangulation changes? Suppose we obtain a new triangulation T' from T by toggling the diagonal E to the opposite diagonal F:

Then we claim that:

- $\tau_\Sigma(F, T') = \tau_\Sigma(E, T)^{-1}$;
- $\tau_\Sigma(A, T') = \tau_\Sigma(A, T)(1 + \tau_\Sigma(E, T)^{-1})^{-1}$;
- $\tau_\Sigma(B, T') = \tau_\Sigma(B, T)(1 + \tau_\Sigma(E, T))$;
- $\tau_\Sigma(C, T') = \tau_\Sigma(C, T)(1 + \tau_\Sigma(E, T)^{-1})^{-1}$;
- $\tau_\Sigma(D, T') = \tau_\Sigma(D, T)(1 + \tau_\Sigma(E, T))$.

In other words, the shear coordinates transform precisely according to the Y-system mutations. Moreover, there is a tropical version of this picture. To see it, fix a *lamination* L on Σ: that is, a collection of pairwise nonintersecting curves satisfying some conditions (like closedness). Then we can define tropical shear coordinates $b_L(T, E)$ for an edge E of our triangulation T by summing up contributions of the curves in L according to the \$/Z(ilch) crossing rule:

The way these tropical shear coordinates $b_L(T, E)$ evolve under the diagonal-flipping toggles is then the same as tropical Y-system mutation: for $k \in [n]$, we define the tropical Y-system mutation μ_k^T by $\mu_k^T(Q, Y) := (Q', Y')$ where $Q' := \mu_k(Q)$ and

$$
Y_j' := \begin{cases} Y_j^{-1} & \text{if } k = j \\ Y_j \left(\frac{Y_k}{(Y_k \oplus 1)}\right)^{-m} & \text{if } k \stackrel{m}{\Rightarrow} j \\ Y_j(Y_k \oplus 1)^m & \text{if } k \stackrel{m}{\Leftarrow} j \end{cases}
$$

Here for two Laurent monomials f and g, we define $f \oplus g$ to be the maximum of the two monomials according to some order like degrevlex. Note that the c-vector (i.e., vector of degrees) associated to this tropical Y-system is the same as the corresponding c-vector for cluster mutations on our quiver. A key observation in Keller's proof [33] is that it is easier to prove periodicity for tropical c-vectors than for the g-vectors of the Y-dynamics itself.

8. Thursday Morning Problem Selection

There was a call for new problems to be suggested, and two were:

8.1. **Darij Grinberg: "Plactoid" monoids.**

One can define plactic-like monoids inside the symmetric group by slightly modifying the Knuth relations and they appear to still behave similarly to the plactic monoid in many respects; for instance, with

$$J := \langle \cdots acb \cdots = \cdots abc \cdots, \cdots bac \cdots = \cdots cab \cdots : a < b < c \rangle$$

we have $\#\mathfrak{S}_n/J = \#\{\text{involutions in } \mathfrak{S}_n\}$. For more on this problem, see [34] [35] [38]. Question: Is there a tableaux theory for these plactoid monoids? Arkady Berenstein asked in particular if we can view $\mathfrak{S}_n/J$ as a Gelfand model: i.e., can we naturally give it the structure of a $\mathfrak{S}_n$-module such that it decomposes into a sum of simple modules where every simple module of $\mathfrak{S}_n$ appears with multiplicity one in this sum.

8.2. **Gregg Musiker: Infinite birational rowmotion.**

As explained in Gregg's lecture, the dynamics of the Y-system are related to the bounded octahedron recurrence [29]. In many respects, the infinite octahedron recurrence [48] is better-behaved than the bounded version. Question: Can we understand birational rowmotion, promotion, or gyration on an "infinite" $\underline{p} \times \underline{q}$ grid by sending $p, q \to \infty$?

After the call for new problems and a summary of group progress from the day before, we voted on the following problems:

(1) The noncrossing partition toggle homomesy problem.

   The group reported that they were still investigating the tantalizing conjecture that any Coxeter toggle is homomesic with respect to number of parts. They also have a conjectural description of the number of fixed points under any τ_{ij} as a simple function of i and j.

(2) The interesting distribution on lattice paths.

   The group is exploring new proofs of the result of [11] as well as extensions to broader classes of partitions. A key step in the proof is the fact that having a left turn at a given position in the grid is equally likely as a right turn, even with respect to this strange distribution. In order to give a new proof of this proposition, the group devised the following collection of toggles on linear extensions $\mathcal{L}(P)$ of a poset P. For $p \in P$ and $w = (w_1, w_2, \ldots, w_i, \ldots, p, \ldots, w_j, \ldots, w_n) \in \mathcal{L}(P)$, where i is the largest index such that $w_i < p$ in P and j is the smallest index such that $p < w_j$ in P, we define $\tau_p(w)$ to be the linear extension we get by reflecting the position of p in w within the interval between w_i and w_j and leaving the relative order of all the other w_k unchanged. The τ_p seem interesting in their own right. One might define a shuffling Markov chain with these toggles as in [2].

(3) The expected number of braid moves in a reduced word in a certain commutation class.

   The group reported that they have translated their problem to the following: find the expected number of times three consecutive entires

appear on the diagonal of a shifted staircase tableau. They conjecture that the action of the group generated by gyration and evacuation (or equivalently, the "even" and "odd" parts of gyration) is homomesic with respect to this statistic. Note that this would be an instance of homomesy for a dihedral group action (as in the question about oscillating tableaux posed by Sam Hopkins above). They also reported that the shifted tableau of trapezoidal is another natural place to study this statistic: in particular, it appears the expectation is now $1/2$ as opposed to 1.

(4) The $3n - 2$ superpromotion resonance problem.

The group reported that they have written some code for studying homomesies and conjecture that ASMs, viewed as order ideals in a tetrahedral poset, are homomesic with respect to cardinality under superpromotion. They also explained that the connection between increasing tableaux and plane partitions suggests that certain symmetry classes of plane partitions may have some explicable resonance phenomena with respect to various toggle group actions: specifically, self-complementary plane partitions should resonate under promotion with a small pseudo-period.

(5) Connections between birational rowmotion and cluster algebras/Y-systems.

The group reported that they will investigate the "$\mathcal{X}$-coordinates" of Harold Williams [61] as they relate to the chamber ansatz.

(6) The $\mathcal{J}(\Phi^+(W)) \to \mathcal{S}_{\mathbf{c}}(W)$ bijection.

The group reported that they do not have a lot of new ideas to attack this difficult problem but plan on gathering more computational data.

(7) The plactoid monoids (as explained by Darij Grinberg above).

(8) Infinite birational rowmotion (as explained by Gregg Musiker above).

All but the last were selected for groups.

9. Friday Morning Lectures

On Friday morning we had two shorter lectures reporting on some of the successes of the week's work.

9.1. **Max Glick: Birational rowmotion and Y-systems.**

First Max reviewed quiver and cluster mutations. Let Q be a quiver on $[n]$. For $k \in [n]$, we define the quiver mutation $\mu_k(Q) := Q'$ where Q' is obtained from Q by:

(1) for each $i \to k \to j$ in Q, add $i \to j$;
(2) reverse all arrows $i \to k$ and $k \to j$;
(3) erase all 2-cycles.

A seed is a pair $(\vec{x}, Q)$ where Q is a quiver on $[n]$ and $\vec{x} = (x_1, \ldots, x_n)$ is a list of rational functions attached to the vertices. For $k \in [n]$, we define the cluster mutation $\mu_k(\vec{x}, Q) := (\vec{x'}, Q')$ where $Q' := \mu_k(Q)$ and

$$x'_j := \begin{cases} x_j, & j \neq k; \\ \frac{\prod_{i \to j} x_i + \prod_{j \to i} x_i}{x_j} & j = k. \end{cases}$$

A famous result of Fomin and Zelevinsky [17] then says the following: fix an initial seed $((x_1, \ldots, x_n), Q)$; then after performing some sequence of mutations, each variable you obtain in the resulting seed can be expressed as a Laurent polynomial in the variables $x_1, \ldots, x_n$. Moreover, these Laurent polynomials conjecturally have nonnegative integer coefficients. (Actually, at the level of cluster mutation defined here, this conjecture has been proven [36]; but there are more general definitions of cluster algebra for which positivity remains a conjecture.) Moreover, the following later result of Fomin and Zelevinsky [18] relates cluster mutation to the Y-mutation explained in Gregg Musiker's talk: let $(\vec{x}, Q)$ be a seed; define $y_j(\vec{x}) := (\prod_{i \to j} x_i)/(\prod_{j \to i} x_i)$; then under cluster mutation of the x_j, the y_j transform according to (birational) Y-mutation. So moving between x and y variables is essentially a change of coordinates (except that the ys may satisfy some relations and thus not uniquely determine the xs).

Next Max reviewed Y-systems and outlined Volkov's proof [59] of periodicty for the $A_r \times A_s$ Y-system. Fix $r, s \geq 1$. The $A_r \times A_s$ Y-system on $\{Y_{i,j,t}\}_{i \in [r], j \in [s], i+j+t \text{ even}}$ is given by

$$Y_{i,j,t+1} Y_{i,j,t-1} = \frac{(1 + Y_{i-1,j,t})(1 + Y_{i+1,j,t})}{(1 + Y_{i,j-1,t}^{-1})(1 + Y_{i,j+1,t}^{-1})}.$$

Here we omit factors that don't make sense (i.e., $1 + Y_{0,j,t} = 1$). The bounded octahedron recurrence is defined by cluster mutations on the same quiver; i.e.,
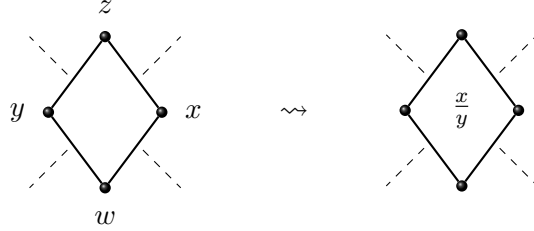
$$x_{i,j,t+1} x_{i,j,t-1} = x_{i-1,j,t} x_{i+1,j,t} + x_{i,j-1,t} x_{i,j+1,t}.$$

For the boundary conditions, we let $x_{0,j,t} = x_{r+1,j,t} = x_{i,0,t} = x_{i,s+1,t} = 1$ (and note that these boundary conditions are apparently different than those of Henriques [29]). Note that the quiver here exhibits a "gyration-esque" phenomenon where if you mutate at all the odd vertices, then mutate at all the even vertices, you get back to where you were. Now recall the theorem Volkov was trying to prove: $Y_{i,j,t+2(r+s+2)} = Y_{i,j,t}$. The outline of Volkov's proof is the following:
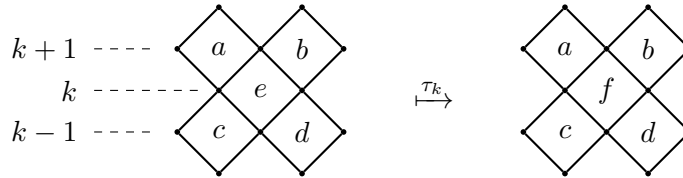
(1) build obviously periodic solutions to the octahedron recurrence (essentially coming from Plücker relations of matrix minors);
(2) use the $x \rightsquigarrow y$ transformation to get periodic solutions to the Y-system;
(3) show that this solution is generic.

Grinberg-Roby's proof [24] of periodicity for birational rowmotion on $P = \underline{p} \times \underline{q}$ was inspired by Volkov's proof. Max went on to explain how birtaional rowmotion on the product of two chains $\underline{p} \times \underline{q}$ and the $A_p \times A_q$ Y-system are indeed formally equivalent. For $k \in [p+q]$, let τ_k be the composition of all birational toggles at rank k. By Grinberg-Roby it follows that $\tau_k \colon \mathbb{R}^P \to \mathbb{R}^P$ descends to a map $\tau_k \colon X \to X$, where X is the set of $\mathbb{R}$-labelings of the poset P modulo rank rescaling. We can therefore employ the

following change of coordinates that moves from variables at the vertices of the Hasse diagram of P to variables at the faces of the Hasse diagram of P:
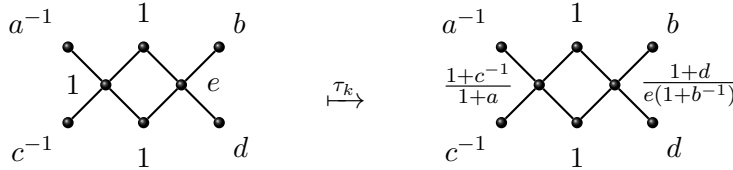

So how does τ_k behave for these homogenized coordinates?

The claim is that

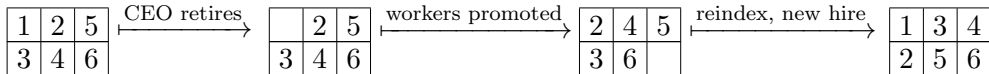$$ef = \frac{(1+a)(1+d)}{(1+c^{-1})(1+b^{-1})}.$$

And the proof of this claim is simple; by rescaling ranks we see that toggling at all elements of rank k does the following:

The upshot is that birational rowmotion does evolve according to Y-system dynamics.

9.2. Oliver Pechenik: Some things that used to bother me and I now understand better.

Oliver explained two, seemingly related, results that he really likes and wants to generalize. The first is the result of Brouwer and Schrijver [7] that rowmotion on $\underline{a} \times \underline{b}$ has order $a + b$. The second is the result attributed to Schützenberger that promotion on $\mathrm{SYT}(a^b)$ has order ab (i.e., the size of the maximal element). Recall the following description of promotion due to Bob Proctor: the rectangular shape represents an office building; there is a beautiful beach on the left side and a trash dump on the right side, so everyone wants to be as far left and high up as possible to have a view of the beach; workers, whose offices are the boxes of the diagram, are ranked according to seniority and report to the people to their left and above them; when someone retires, people adjacent to them fill their spot according to seniority and then ranks are reindexed and a new hire is made to fill the empty spot. Promotion is what happens when the CEO, numbered 1, retires:

How can we generalize these two results? Cameron and Fon-Der-Flaass [8] study rowmotion on $\underline{a} \times \underline{b} \times \underline{c}$, and specifically give evidence that the order resonates with pseudo-period $a + b + c - 1$, which means roughly that the order "cares" about this number. Recall that the set of order ideals $\mathcal{J}(\underline{a} \times \underline{b} \times \underline{c})$ of the product of three chains is the same as the set of plane partitions inside a $a \times b \times c$ box. For an example of this resonance phenomenon, consider the following plane partition inside $4 \times 4 \times 4$:

4	4	4	3
4	3	2	2
3	2	1	1
3	1	0	0

.

That plane partition has period 33 under rowmotion, which is $3(a+b+c-1)$. And how can we generalize promotion on SYT? Pechenik [40] defined and studied k-promotion on increasing tableaux $\mathrm{Inc}(a^b, k)$. Recall that the tableaux in $\mathrm{Inc}(\lambda, k)$ are fillings of the boxes by λ by elements of $[k]$ such that entries strictly increase in both rows and columns. Maintaining the office analogy, the rules for k-promotion on increasing tableaux are that times are tough, so you can hold two jobs at once, but you can never be your own boss; the local rules are:
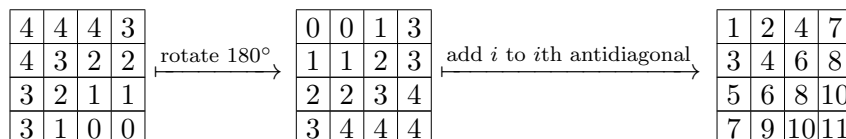

Pechenik [40] gave evidence that that the order of k-promotion on increasing tableaux $\mathrm{Inc}(a^b, k)$ resonates with k. For example, with $k = 1$, we have that the period of the increasing tableaux

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

is $33 = 3k$.

The amazing discovery made during this workshop by Kevin Dilks, Oliver Pechenik, and Jessica Striker is that these two mysterious resonances are in fact the same mystery. Specifically, we have a bijection $\Psi \colon \mathcal{J}(\underline{a} \times \underline{b} \times \underline{c}) \to \mathrm{Inc}(a^b, a + b + c - 1)$ given by the composition of the following two simple procedures:

From this description it should be clear that Ψ is bijective. But, in fact Ψ is equivariant! That is, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{J} & \xrightarrow{\ \Psi\ } & \mathrm{Inc} \\
\text{promotion}\downarrow & & \downarrow k\text{-promotion} \\
\mathcal{J} & \xrightarrow[\ \Psi\ ]{} & \mathrm{Inc}
\end{array}$$

Here "promotion" on $\mathcal{J}(\underline{a} \times \underline{b} \times \underline{c})$ is in the sense of Striker-Williams [54]. Recall that Striker and Williams show that promotion and rowmotion on order ideals are conjugate in the toggle group and thus have the same orbit structure. So Ψ indeed explains why rowmotion on order ideals in the product of three chains and k-promotion of increasing tableaux exhibit the same resonance phenomenon.

Moreover, from the symmetric role of a, b, and c in $\mathcal{J}(\underline{a} \times \underline{b} \times \underline{c})$, we actually get *three* equivariant bijections:

$$\Psi_1 \colon \mathcal{J}(\underline{a} \times \underline{b} \times \underline{c}) \to \mathrm{Inc}(a^b, a + b + c - 1)$$
$$\Psi_2 \colon \mathcal{J}(\underline{a} \times \underline{b} \times \underline{c}) \to \mathrm{Inc}(a^c, a + b + c - 1)$$
$$\Psi_3 \colon \mathcal{J}(\underline{a} \times \underline{b} \times \underline{c}) \to \mathrm{Inc}(b^c, a + b + c - 1).$$

Composing these bijections and symmetries yields many interesting results "for free." For example, when $c = 1$, Brouwer-Schrijver [7] show that rowmotion has order $a + b$ (actually, to be more accurate we should say that rowmotion to the $a + b$ is the identity, but we will ignore the distinction between the order being n and the order dividing n for all these maps). Via Ψ_2 this $c = 1$ result becomes a trivial statement; k-promotion on increasing tableaux of one row clearly has order k. So Ψ_2 gives a new proof of [7]. Via Ψ_1 this $c = 1$ result becomes a new result about k-promotion on $\mathrm{Inc}(a^b, a + b)$ that Oliver had earlier conjectured but could not prove. Similarly, when $c = 2$, Cameron and Fon-Der-Flaass [8] show that rowmotion again has order $a + b + 1$. Pushed through Ψ_2 this $c = 2$ case recovers results of Pechenik [40] (or you could say that [40] plus Ψ_2 gives a new proof of [8]). And pushed through Ψ_1, this $c = 2$ case yields a new result about k-promotion on $\mathrm{Inc}(a^b, a + b + 1)$ that again Oliver conjectured but could not prove up until now. Oliver went on to conjecture that in the $c = 3$ case we still have that the order of rowmotion on $\mathcal{J}(\underline{a} \times \underline{b} \times \underline{c})$ is (or divides) $a + b + 2$.

Oliver then explained how these equivariant bijections can give us a more formal way of understanding resonance. For $P \in \mathcal{J}(\underline{a} \times \underline{b} \times \underline{c})$, define the content of P to be the $3 \times (a + b + c - 1)$ $0, 1$ matrix

$$\begin{pmatrix}
\chi_1(\Psi_1 P) & \chi_2(\Psi_1 P) & \cdots \\
\chi_1(\Psi_2 P) & \chi_2(\Psi_2 P) & \cdots \\
\chi_1(\Psi_3 P) & \chi_2(\Psi_3 P) & \cdots
\end{pmatrix}$$

where $\chi_n(T)$ for an increasing tableaux T is the indicator function that is 1 if T contains an n and 0 otherwise. A corollary of the theorem that these Ψ are equivariant bijections is that promotion (in the Striker-Williams sense) on a plane partition P descends to cyclic rotation of the content of P. Note that a generic content matrix has no symmetries, so a generic orbit of promotion has to have order divisible by $a + b + c - 1$. This content matrix led Oliver to propose the following definition of resonance: we say a cyclic action $\mathbb{Z} \curvearrowright X$ *resonates* with f if there is an equivariant map $X \to Y$ where $\mathbb{Z}$

acts on Y by rotation by $\frac{2\pi}{f}$ with at least one orbit of size f. Here we are supposed to view Y as a geometric object that is literally rotated. Much like cyclic sieving or homomesy, this definition is meta-mathematical in the sense that we really desire a *natural* equivariant map $X \to Y$.

10. Updates from after the workshop

- The **interesting distribution on lattice paths** group wrote a paper [10] extending the harmonic mean result of Chan et al. [11] to a much broader class of distributions and skew shapes. They also found a connection between this problem and homomesy for the antichain cardinality statistic for rowmotion and gyration.
- The **expected number of braid moves** group wrote a paper [47] proving Williams's conjecture. They also put forward an interesting homomesy conjecture involving a nonabelian (dihedral) group.

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