

Promotion, webs, and plabic graphs

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[Howard University]

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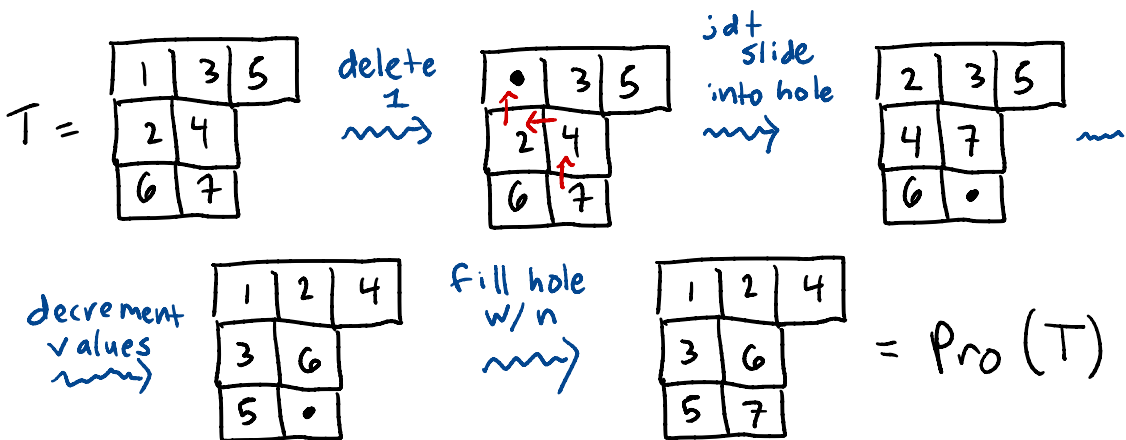
BIRS Workshop on Dynamical Algebraic
Combinatorics

(sort of) based on joint work w/ Martin Rubey
arXiv:2005.14031 [TU Wien]

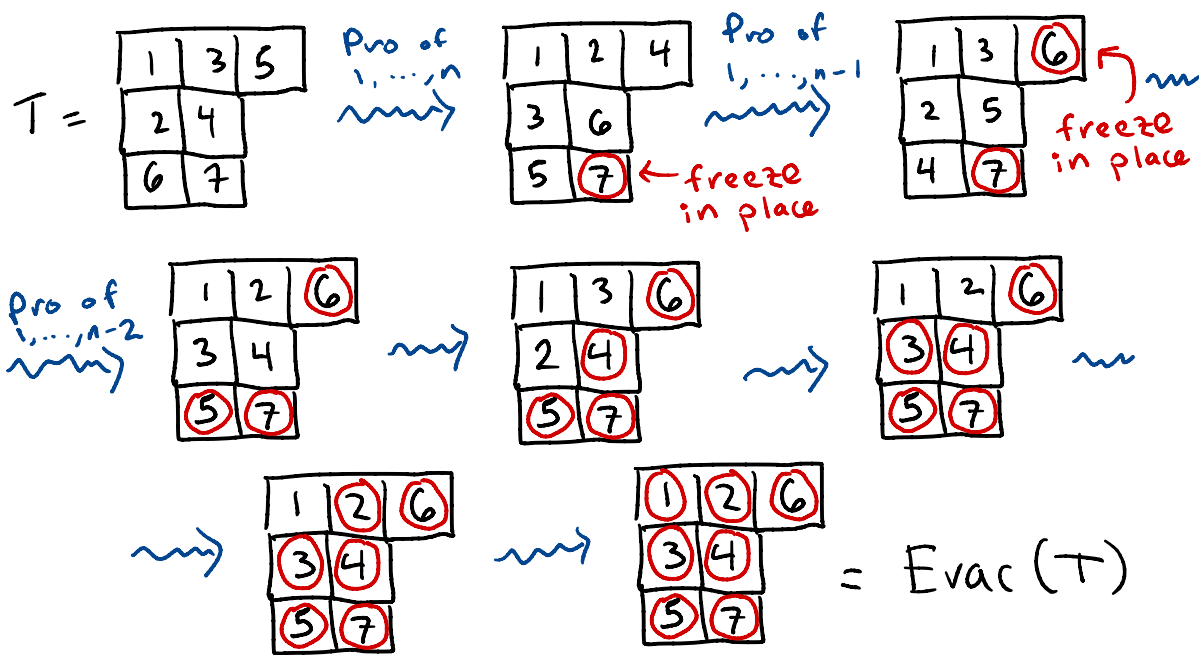
these slides are on my website:
samvelshopkins.com

(+ this talk is being recorded)

Promotion is a cyclic action on the set of Standard Young tableaux (SYTs) of shape $\lambda \vdash n$:



Introduced by Schützenberger, together w/ related operation of evacuation:



While $\boxed{\text{Evac}^2(T) = T}$ for any T , the behavior of Pro is in general hard to predict, except for special shapes λ :

Thm (Schützenberger)

For $\lambda = a \begin{array}{|c|c|c|c|} \hline & b & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$ (the $a \times b$ rectangle),

$\boxed{\text{Pro}^{ab}(T) = T}$ for all T .

Want to study Pro (+ Evac) via models:

Thm (Rhoades)

"long cycle"
↓

For $\lambda = a \times b \vdash n$, the action of $c = (1, 2, \dots, n)$ on G_n ^{← sym. gp.}-irrep S^λ corresponds* to Pro in the KL cellular basis.

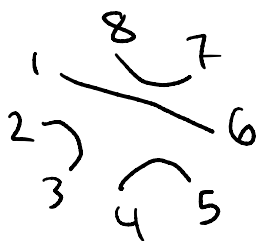
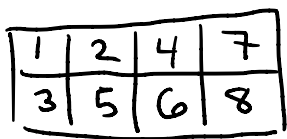
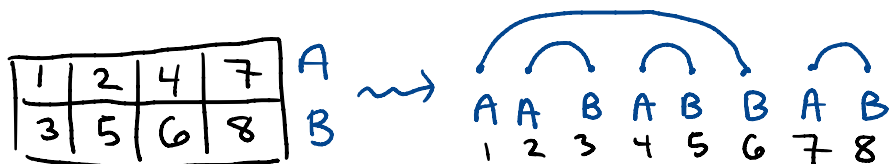
Thm (Stembridge) For any $\lambda \vdash n$, the action of $w_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$ on S^λ corresponds* to Evac in KL cell. basis.

"longest word"
↙

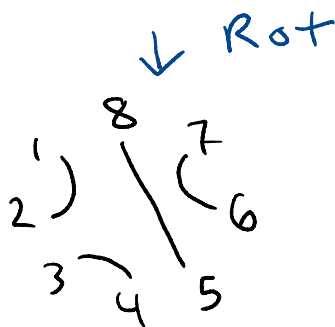
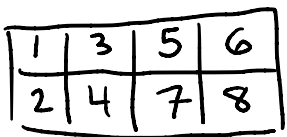
These algebraic models are nice,
but what about **combinatorial** models?

Prop. (D. White)

Promotion of $2 \times n$ SYTs corresponds
to rotation of noncrossing matchings of $[2n]$
via **standard bijection** between
these Catalan objects:



Pro $\downarrow$



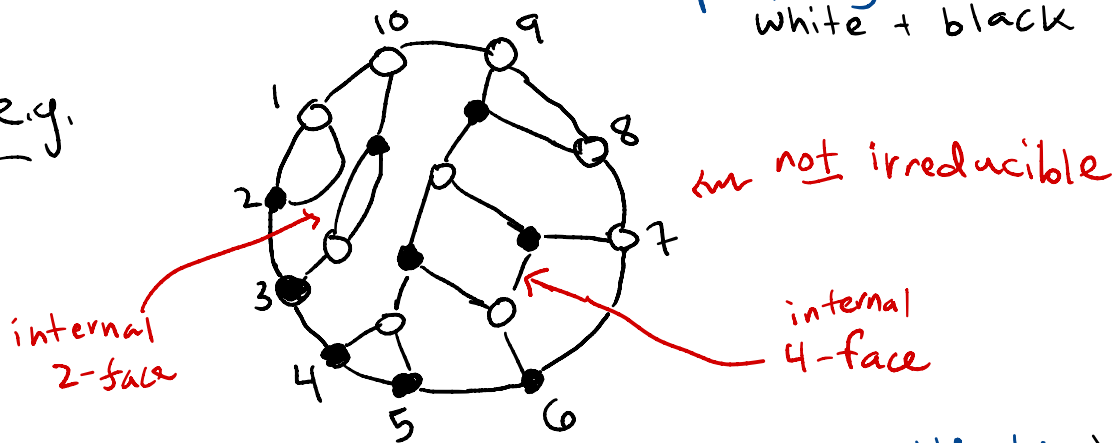
What about more than 2 rows...?

Def'n (Kuperberg)

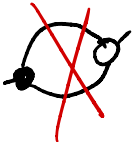
An (sl_3-) web is a planar graph drawn in a disc w/ boundary vertices $1, 2, \dots, n$, and some internal vertices, such that:

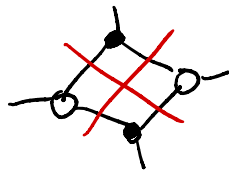
- bdy. vertices have degree one,
- int. vertices are trivalent,
- vertices (bdy. + int.) are properly 2-colored white + black

e.g.



Web is called irreducible (or non-elliptic) if it has no internal (i.e., disjoint from bdy.)

2-faces  and 4-faces



Thm (Khovanov-Kuperberg,
 Petersen-Plyavsky-Rhoades, Tymoczko)

There is a **bijection**
 $\{3 \times n \text{ SYTs}\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{irred. webs } w \\ 3n \text{ white bdy. vert's} \end{array} \right\}$

such that $\text{Pro of } T = \text{Rot of } W$.



How does **bij.** work? Uses M-diagram:

$T =$

1	2	6
3	4	8
5	7	9

w

x

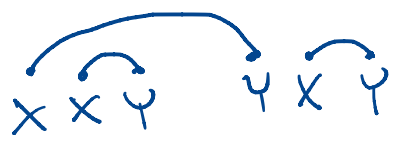
y

$\rightsquigarrow$ $w \ w \ x \ x \ y \ w \ y \ x \ y$
 1 2 3 4 5 6 7 8 9

noncross. match
 of w 's + x 's:

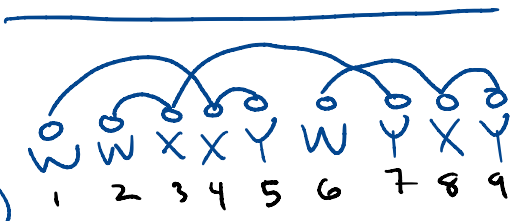


noncross. match
 of x 's + y 's:

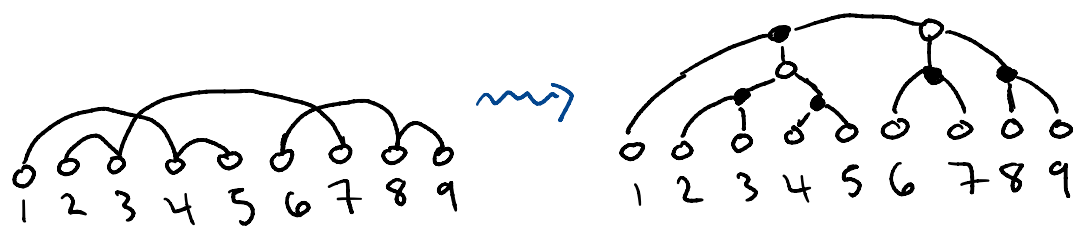
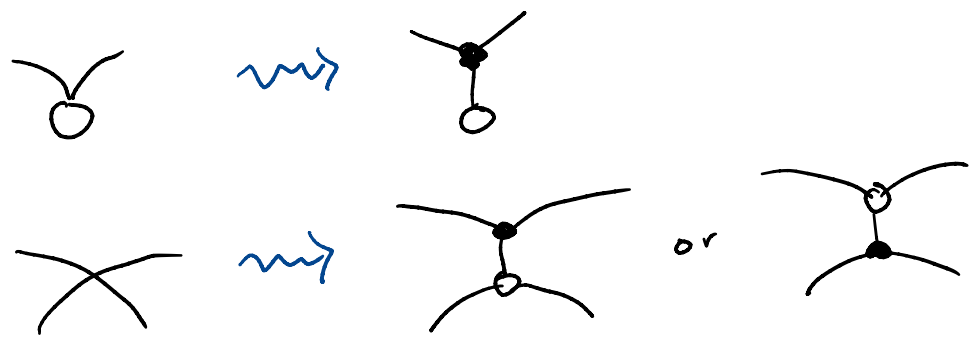


Super-impose
 to get M-diagram:

(compare w/ 'noncrossing tableaux', of Pylyavsky)

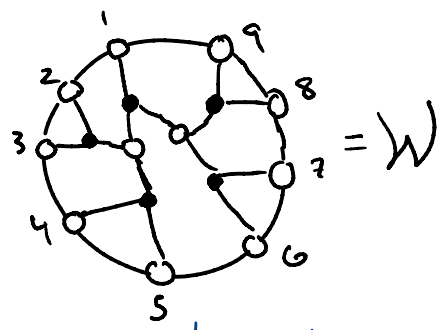


Have to resolve crossings to get web:



$T =$

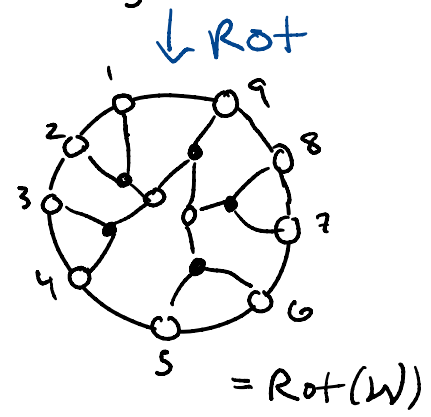
1	2	6
3	4	8
5	7	9



Pro ↓

$Pro(T) =$

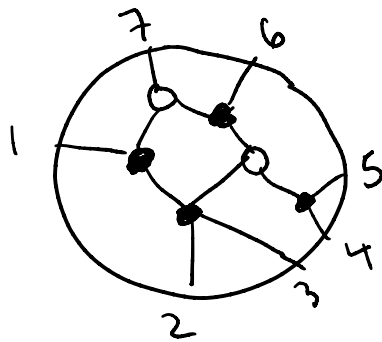
1	3	5
2	6	7
4	8	9



Def'n (Postnikov)

A plabic graph is a planar graph, drawn in a disc, w/ $\deg = 1$ bdy. Vert's $1, 2, \dots, m$, and some int. vert's, such that int. vert's are colored black + white.
 $\nearrow$ not necessarily properly!

eg.:

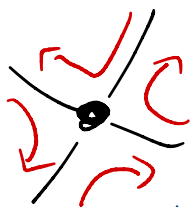


Observation: A web is (basically) a special case of a plabic graph!

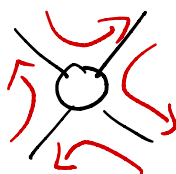
Q: What does this observation buy us?

Def'n (Postnikov)

The trip permutation^{*} of a plabic graph has $\pi(i) =$ bdy. vertex you end at if you start a trip at i and follow the rules of the road:

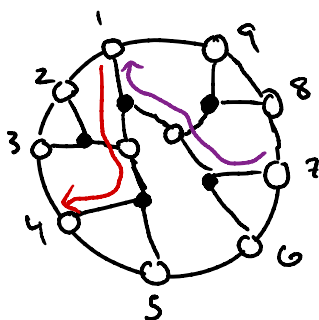


right at black



left at white

e.g.

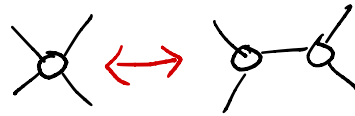
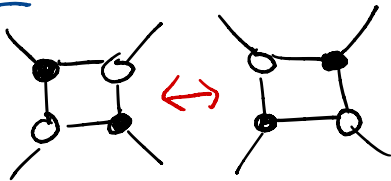


trip starting at 1 ends at 4
trip starting at 7 ends at 1
etc ...

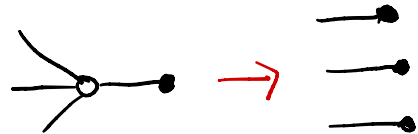
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 8 & 5 & 2 & 7 & 1 & 9 & 6 \end{pmatrix}$$

Def'n (Postnikov) Moves and reductions are local transformations of plabic graphs:

Moves:



Reductions:



Write $G \sim G' :=$ G' can be obtained from G via a series of moves.

Say G is reduced if there is no $G' \sim G$ s.t. can apply a reduction to G' .

Thm (Postnikov) G, G' reduced plabic graphs have same trip perm. $\Leftrightarrow G \sim G'$.

Cor Two irreducible webs W, W' have
same trip perm. $\Leftrightarrow^* W = W'$.

Pf: Only moves/reductions we can
do are trivial: $\text{---} \leftrightarrow \text{---} \circ$ 

For $3 \times n$ SYT T , let Π_{trip} be
the trip perm. of associated web W .

So T is determined by Π_{trip} .

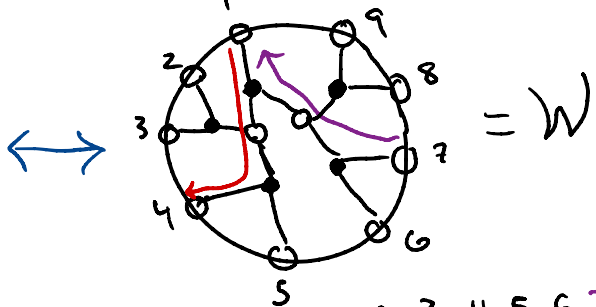
Q: Another way to find Π_{trip} ?

A: YES! Using promotion:

$\Pi_x(i) \equiv$ the # that moves up
from 2nd row when $+ (i-1) \pmod{3n}$
applying Pro to
 $\text{Pro}^{i-1}(T)$

Π_y = Same but w/ 3rd row instead of 2nd

e.g.

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array}$$


$$\pi_{trip} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 8 & 5 & 2 & 7 & 1 & 9 & 6 \end{pmatrix}$$

$T =$

1	2	6
3	4	8
5	7	9

Pro. →

1	3	5
2	6	7
4	8	9

→

1	2	4
3	5	6
7	8	9

Below the first table: $4+0=4$ (red), $7+0=7$ (purple)

Below the second table: $2+1=3$ (red), $4+1=5$ (purple)

Below the third table: $6+2=8$ (red), $9+2=2$ (purple)

→

1	3	5
2	4	8
6	7	9

→

1	2	4
3	6	7
5	8	9

→

1	3	6
2	5	8
4	7	9

Below the first table: $2+3=5$ (red), $7+3=1$ (purple)

Below the second table: $7+4=2$ (red), $9+4=4$ (purple)

Below the third table: $2+5=7$ (red), $4+5=9$ (purple)

→

1	2	5
3	4	7
6	8	9

→

1	3	4
2	6	8
5	7	9

→

1	2	3
4	5	7
6	8	9

Below the first table: $4+6=1$ (red), $8+6=5$ (purple)

Below the second table: $2+7=9$ (red), $5+7=3$ (purple)

Below the third table: $7+8=6$ (red), $9+8=8$ (purple)

$$\pi_X = 438527196 = \pi_{trip} \quad \pi_Y = 752149538 = \pi_{trip}^{-1}$$

Thm For any $3 \times n$ SYT T ,

(a) $\pi_X = \pi_{\text{trip}}$,

(b) $\pi_Y = \pi_{\text{trip}}^{-1}$.

Pf! For (a), first note that π_{trip} , π_X both rotate $\xleftarrow{w} \rightarrow cwc^{-1}$ when we apply Pro to T .

So suffices to prove $\pi_{\text{trip}}(1) = \pi_X(1)$.

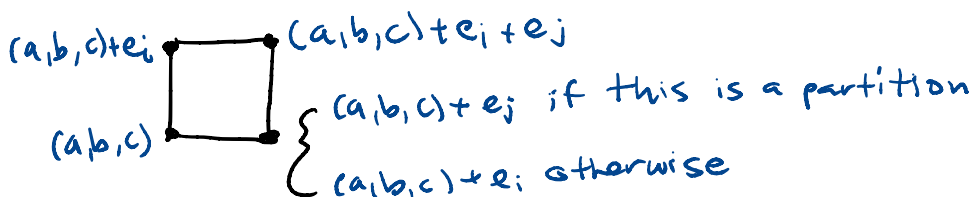
But it's easy to see both
= what 1 is matched with in M -diagram.

For (b), follows by considering dual SYT. ↖ 180° rot. □

Rmk: Representing T via a chain of partitions $\lambda^0 \lambda^1 \lambda^2 \dots \lambda^{3n-1} \lambda^{3n}$ in usual way,

can read off π_X , π_Y in a beautiful way
from the growth diagram of T .

Rule for growth diagram: (see Fomin, Roby, Stanley...)

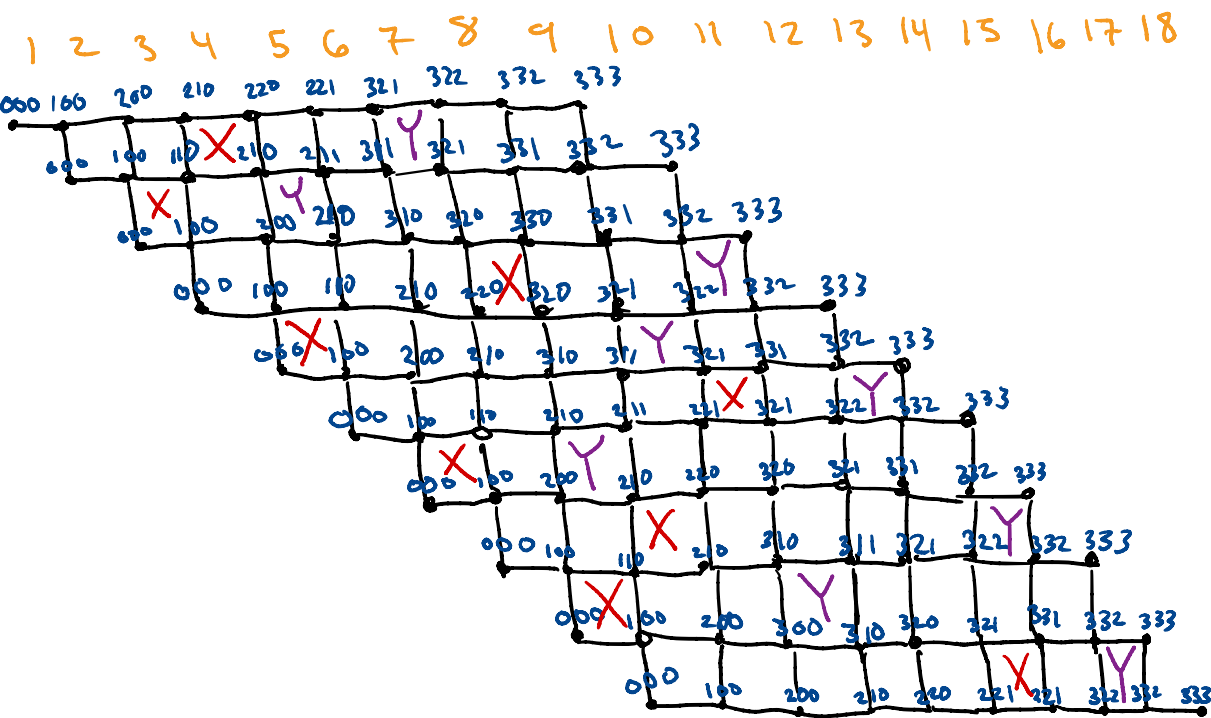


↳ In this 2nd case mark box w/

X if $j=2$

Y if $j=3$

Growth diagram for full promotion orbit of T:



$$\pi_X = 4, 3, 8, 5, 11, 7, 10, 9, 15 = 4, 3, 8, 5, 2, 7, 1, 9, 6$$

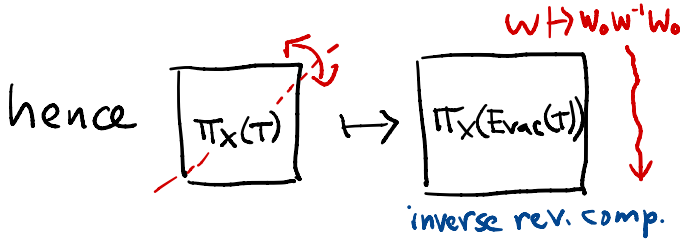
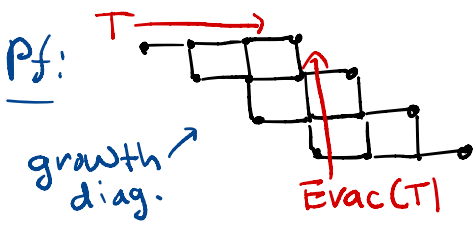
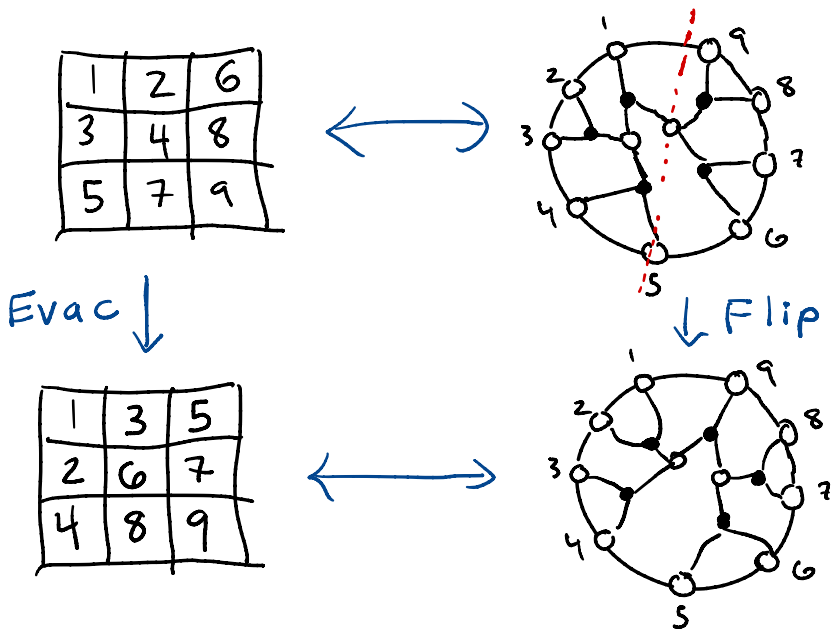
$$\pi_Y = 7, 5, 11, 10, 13, 9, 15, 12, 17 = 7, 5, 2, 1, 4, 9, 6, 3, 8$$

So... what? What can we do w/ this?
 Well... I'd love you to tell me!

But here is one thing I know we can do!

Cor (c.s. Patrias-Pechenik) $\text{Evac}(T) \leftrightarrow \text{Flip}(W)$,
 where $\text{Flip}(W) = \text{refl. of } W \text{ across a diam.}$

e.g.



But since trip perm. of $\text{Flip}(W) = \text{inv. r.c. of trip perm. of } W$,
 $\text{Flip}(W)$ is only possibility for $\text{Evac}(T)$. $\square$

But... what else?

- do this in other, similar contexts:
e.g., semistandard $3 \times n$ tableaux (Russell)
or linear extensions of $VX[n]$ (H.-Rubey)
or $a \times b$ SYT w/ $a > 3$ (ambitious!)
- Can we classify perm's of form Π trip,
say for $3 \times n$ SYTs?
- Well-known that we can either make a
(red.) plabic graph trivalent or bipartite
via moves. Webs are when we can do both.

Any significance to that?

- Plabic graphs + Webs both related to the
Grassmannian $Gr(k, n)$:
 - i) Plabic graphs parameterize cells in the
totally nonnegative Grass. $Gr_{\geq 0}(k, n)$,
 - ii) Webs yield tensor invariants, which can
be viewed as elements of $\mathbb{C}[Gr(k, n)]$.
- Is there a direct connection (c.f. Fraser-Lam-Le)?
Where does promotion fit in?

Thank you!

(+ hope to see you all in person
Sometime soon ...
maybe in DC?)

