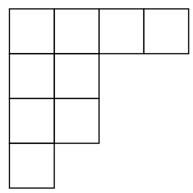
CRUX – THE LAST PROBLEM: DEMYSTIFIED

SAM HOPKINS

The June 2021 issue of *Crux Mathematicorum* included, on page 287, an intriguing problem entitled simply "The Last Problem." Here we aim to demystify this problem, and, if not exactly explain its solution, at least situate it in its proper mathematical context.

A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. For example, $\lambda = (4, 2, 2, 1)$ is a partition of 9. Associated to any partition is its *Young diagram*: the left- and top-justified array of boxes which has λ_i boxes in the ith row. The Young diagram of $(4, 2, 2, 1)$ is



A *standard Young tableau (SYT)* of shape λ is a filling of the Young diagram of λ with the numbers $1, 2, \ldots, n$ so that these numbers are strictly increasing along rows and down columns. An SYT of shape $(4, 2, 2, 1)$ is

1	3	5	8
2	4		
6	9		
7			

Young diagrams and tableaux are named after Alfred Young (1873–1940), a British mathematician who pioneered the study of the group of permutations of a finite set. Much is known about SYTs because of their connection to algebra. For instance, there is a beautiful formula for the number of SYTs of given shape. The *hook* of a box u in a Young diagram consists of all boxes directly below, or directly to the right of, that box, including the box itself. The *hook length* of a box is the number of boxes in its hook. For example, the box with entry 2 in the above SYT has a hook length of 4. The celebrated *hook length formula* says that the number of SYTs of shape λ, a partition of n, is

$$\frac{n!}{\prod_u h(u)},$$

where the product runs over all boxes u of the Young diagram of λ, and $h(u)$ is the hook length of the box u. So for example there are $9!/(7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1) = 216$ SYTs of shape $(4, 2, 2, 1)$. To learn more about tableaux in general, see Yong's short note [3] or Sagan's survey [1].

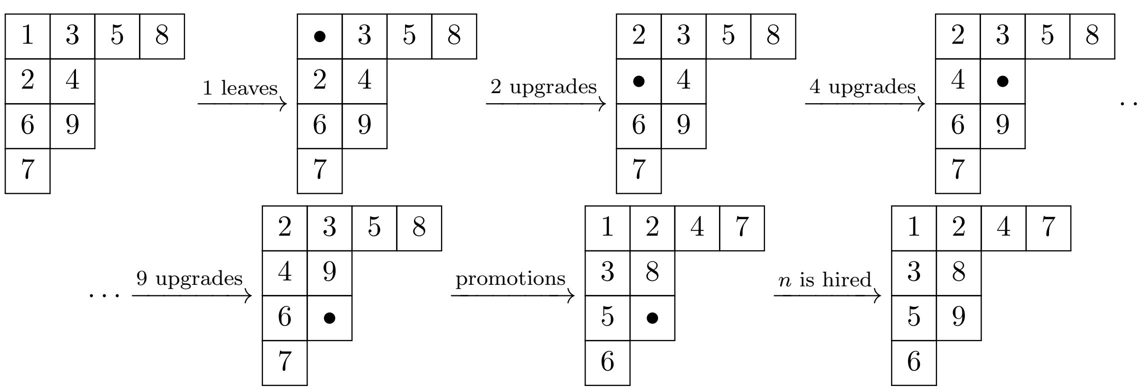
Date: September 28, 2021.

Our present interest in SYTs lies not in their enumeration but rather in a certain operation on them, which we now explain using a somewhat fanciful analogy.

We can view a Young diagram as a building whose rooms are the boxes of the diagram. (This presents some engineering challenges because the rows get longer towards the top, but never mind that.) If this building belongs to a hierarchical organization, like a company, then we can view a filling of the Young diagram with numbers $1, 2 \ldots,$ as an assignment of rooms to the person of rank 1, the person of rank 2, and so on. Suppose that to the left of our Young diagram office building is a beautiful ocean. Naturally, everyone in the building wants to have the best view of this ocean, and hence would always prefer to have a room as much to the left (to be closer to the ocean) and above (to have a higher viewpoint) as possible. So, in order to respect the pecking order, we might require room assignments to be such that every person has a lesser rank than the people in the rooms to their left and above them. Room assignments like this are precisely SYTs.

But now suppose that the CEO (the person of rank 1) leaves the company for a better opportunity elsewhere. Their departure creates an opening in a very desirable room. Those of lesser rank will fill this opening. Since the company does not want people moving their stuff a long way across the building, only the people whose rooms are adjacent, either to the right or below, can compete to fill that open room. Of course, among these two, the room is awarded to the person of greater rank. They move from their current room to the more desirable one, and in doing so they create a new room opening, which is filled in the same manner: with the people currently adjacent to the right and below competing. In this way the departure of the CEO causes a series of room re-assignments, which eventually terminates with an undesirable room at the bottom-right of the building being emptied. Then, two final things happen to complete the corporate restructuring. First, everyone in the building gets a "promotion," meaning that the person of rank 2 becomes rank 1, the person of rank 3 becomes rank 2, and so on. And second, a new intern, of the least rank n, gets hired to fill the empty room.

Here is an example of this procedure:



This entire operation, which takes one SYT of shape λ to another one, is in fact called *promotion*. The promotion operation on tableaux was introduced, together with another closely related operation called *evacuation*, by M.P. Schützenberger.[1] The sliding process which goes into the definition of both promotion and evacuation was termed *jeu de taquin*

[1] The French mathematician Marcel-Paul Schützenberger (1920–1996) had a wide range of scientific interests: e.g., he obtained a doctorate in medicine in 1948; and in the 1960s he worked with the famous linguist Noam Chomsky on the analysis of formal languages. In algebraic combinatorics he is especially remembered for seminal contributions to the theory of tableaux, symmetric functions, and Schubert calculus.

by Schützenberger. "Jeu de taquin" literally translates to "teasing game," but is the name in French for what is usually called the "15 Puzzle" in English. For an excellent introduction to promotion and evacuation, see Stanley's survey [2].

Promotion is an invertible operation. To see this, we can imagine doing all of the steps backwards: firing the intern, demoting everyone, and forcing them into worse rooms until there is a spot at the top for a new CEO. Hence, for any given tableau T there must be some number of times we can apply promotion to T that will get us back to T. But for most shapes, promotion behaves quite chaotically and it takes a long time for us to get back to where we started. For instance, we would need to apply promotion 60 times to our running example SYT of shape $(4, 4, 2, 1)$ in order to return to it. (In contrast, evacuation is always an involution, meaning if we apply it twice we get back to where we started.)

There are a very small number of partition shapes for which promotion behaves in an orderly fashion: see [2, §4]. These nice shapes include the *rectangle*[2] $a \times b := (\overbrace{b, b, \ldots, b}^{a \text{ times}})$. For any SYT of rectangular shape $a \times b$, if we apply promotion ab times we get back to where we started; this is [2, Theorem 4.1(a)]. Note that we might get back to where we started even before ab applications of promotion. For example, for the following SYT of shape 2×3, we can apply promotion 3 times to return to it:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \xrightarrow{\text{Promotion}} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{Promotion}} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \xrightarrow{\text{Promotion}} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$$

By now the reader may recognize that "The Last Problem" from the June 2021 issue of *Crux* precisely concerns the promotion operation applied to SYTs of rectangular shape.[3] Part (a) of the problem asks the reader to show that ab applications of promotion applied an SYT T of shape $a \times b$ returns the initial tableau T. There is no really simple proof of this fact: it does follow from the fundamental properties of jeu de taquin as developed by Schützenberger, but it takes quite a while to develop this theory. We would be very impressed if any reader submitted a correct solution to this problem. Parts (b) and (c) also follow from known properties of promotion: see [2, Theorem 2.3], which relates the "principal chain" and "trajectory" of a tableau, and [2, Theorem 4.1(a)], which explains that evacuation for rectangular SYTs is $180°$ rotation plus swapping each number i for $n + 1 - i$.

Although we were not able to explain the full solution to "'The Last Problem" in this short space, we hope that we have inspired the reader to learn more about tableaux and their fascinating properties, as well as the dynamical operations defined on them.

References

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[2] R. P. Stanley. Promotion and evacuation. *Electron. J. Combin.*, 16(2):Paper 2.9, 24, 2009. Available online here.

[3] A. Yong. What is ... a Young tableau? *Notices Amer. Math. Soc.*, 54(2):240–241, 2007. Available online here.

E-mail address: `samuelfhopkins@gmail.com`

Department of Mathematics, Howard University, Washington, DC

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[2]Shapes that behave well under promotion also include the *staircase* $\delta_n = (n, n-1, \ldots, 1)$, though understanding promotion for the staircase is even more involved than for the rectangle.

[3]But note that the arrays there are $180°$ rotations of SYTs as we defined them here.