Bigraphical arrangements

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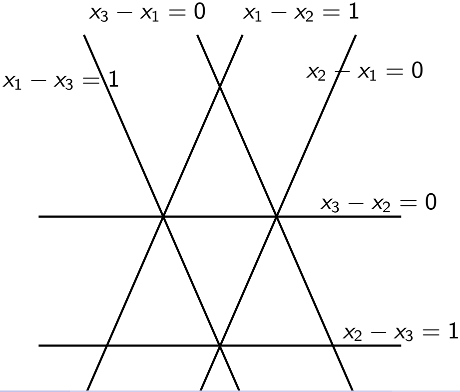
The Shi arrangement

Jian Yi Shi in his 1986 study of the Kazhdan-Lusztig cells of the affine Weyl group $\widetilde{A}_{n-1}$ defined (what is now called) the *Shi arrangement*:

$$\mathrm{SHI}(n) := \{x_i - x_j = 0, 1\colon 1 \le i < j \le n\}.$$

Shi showed that the number of regions is $r(\mathrm{SHI}(n)) = (n+1)^{n-1}$.



Stanley's problem

From Cayley, we know that $(n+1)^{n-1}$ counts the number of spanning trees of the (labeled) complete graph K_{n+1}.

Problem (Stanley)

Find a bijection between regions of $\mathrm{SHI}(n)$ *and spanning trees of* K_{n+1}.

Circa 1994, Pak came up with such a bijection using parking functions (for which many bijections with spanning trees exist). Stanley proved that this bijection works, and the procedure has come to be called the "Pak-Stanley labeling" of the Shi arrangement.

Parking functions

Definition

A *parking function* of length n is a sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ of nonnegative integers such that its weakly increasing rearrangement $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_n}$ satisfies $(a_{i_1}, \ldots, a_{i_n}) \leq (0, 1, \ldots, n-1)$.

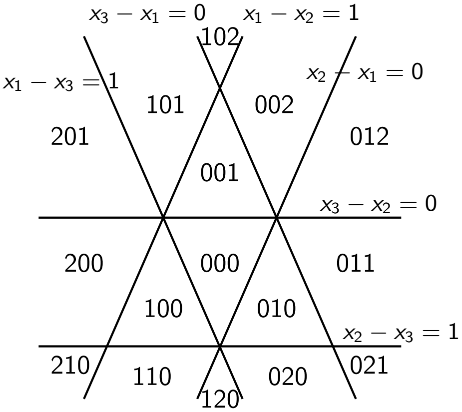
The name "parking function" comes from a certain amusing description due to Knuth, that unfortunately I'm going to skip, in terms of cars trying to park on a one-way street.

Example

The parking functions of length 3 are

$$(0,0,0),(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(0,0,2)$$
$$(0,2,0),(2,0,0,),(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0)$$

The Pak-Stanley labeling



Pak-Stanley labeling: start by labeling central region ($0 < x_i - x_j < 1$) with $(0, 0, \ldots, 0)$. Inductively label adjacent regions:

- cross a hyperplane of the form $x_i - x_j = 0 \Rightarrow$ increase jth coordinate;
- cross a hyperplane of the form $x_i - x_j = 1 \Rightarrow$ increase ith coordinate.

Graphical parking functions

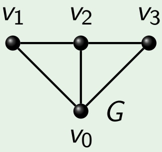
$G = (V, E)$ is a (finite, simple, connected) graph with $V = \{v_0, \ldots, v_n\}$. Designate v_0 to be the special *sink* vertex. Set $\widetilde{V} := V \setminus \{v_0\}$.

Definition

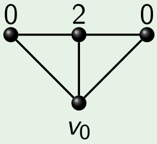
A *G-parking function* with respect to v_0 is $c = \sum_{i=1}^{n} c_i v_i \in \mathbb{Z}\widetilde{V}$ such that for all $U \subseteq \widetilde{V}$, there exists some $v_i \in U$ such that $0 \leq c_i < d_U(v_i)$; where for $u \in U$ we define

$$d_U(u) := |\{\{u, v\} \in E \colon v \in V \setminus U\}|.$$

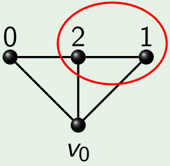
Example



Then



is a G-pf but



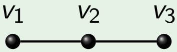
is not.

The G-Shi arrangement

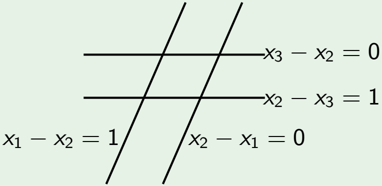
Now $G = (V, E)$ is a (finite, simple) graph with $V = \{v_1, \ldots, v_n\}$.
In 2011, Duval-Klivans-Martin, motivated by their ongoing work extending sandpile theory to higher dimension, defined the G-Shi arrangment

$$\mathrm{SHI}(G) := \{x_i - x_j = 0, 1\colon \{v_i, v_j\} \in E \text{ with } i < j\}.$$

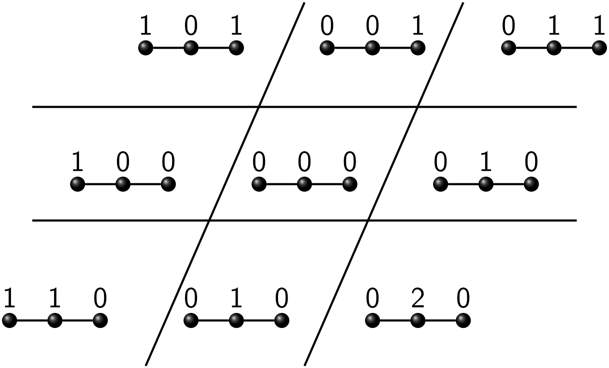
Example

Let G be 

. Then $\mathrm{SHI}(G)$ is



Duval-Klivans-Martin's probelm

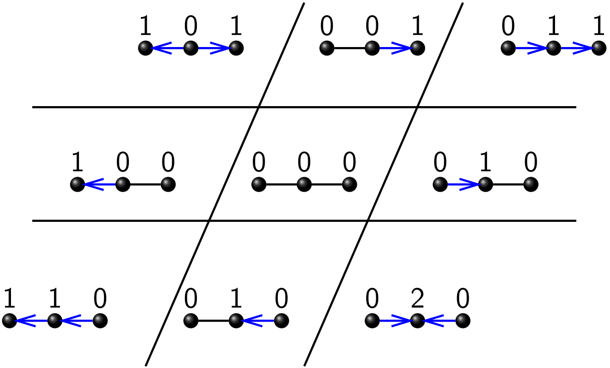


Let G° denote the graph G with an extra vertex v_0 connected by an edge to each vertex in G. We always take v_0 to be the sink of G°.

Problem (DKM)

Show that $\{\textit{Pak-Stanley labels of } \mathrm{SHI}(G)\} = \{\textit{parking functions of } G^\circ\}$.

Labeling G-SHI regions with partial orientations



A partial orientation $\mathcal{O}$ of G is where we orient some edges of G and leave others blank. We identify $\mathcal{O}$ with its set of oriented edges; i.e. $\mathcal{O} \subseteq V^2$. Partial orientations label the regions of $\mathrm{SHI}(G)$ in an natural way shown above. The map $\mathcal{O} \to \mathrm{indeg}(\mathcal{O})$ recovers Pak-Stanley labels.

SHI-admissible partial orientations

When does a partial orientation appear as the label of a region of $\mathrm{SHI}(G)$?

Definition

A *potential cycle* C of $\mathcal{O}$ is an oriented cycle of G such that if $(v_i, v_j) \in C$ then $(v_j, v_i) \notin \mathcal{O}$: that is, we can only walk along blank edges, or oriented edges in the right way. We give C a score $\nu_{\mathrm{SHI}}(C)$:

$$\nu_{\mathrm{SHI}}(C) := \#\{(v_i, v_j) \in C, i < j \colon \{v_i, v_j\} \text{ is blank in } \mathcal{O}\} \\ - \#\{(v_j, v_i) \in C, i < j \colon (v_j, v_i) \text{ is oriented in } \mathcal{O}\}.$$

We say $\mathcal{O}$ is SHI-admissible if $\mathcal{O}$ appears as a label of a region of $\mathrm{SHI}(G)$.

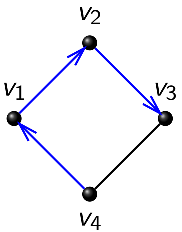
Theorem

$\mathcal{O}$ is SHI-admissible iff every potential cycle of $\mathcal{O}$ has positive score.

Pf: Farkas' lemma.

Note: this forces $\mathcal{O}$ to be acyclic.

SHI-admissible partial orientation example



is not SHI-admissible because there is the potential cycle $C = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ and

$$\nu_{\mathrm{SHI}}(C) = \#\{(v_3, v_4)\} - \#\{(v_4, v_1)\} = 0.$$

Indeed, the oriented edge (v_1, v_2) gives $x_1 < x_2$, the oriented edge (v_2, v_3) gives $x_2 < x_3$, the blank edge $\{v_3, v_4\}$ gives $x_3 < x_4 + 1$, and the oriented edge (v_4, v_1) gives $x_4 + 1 < x_1$. Summing these inequalities leads to a contradiction.

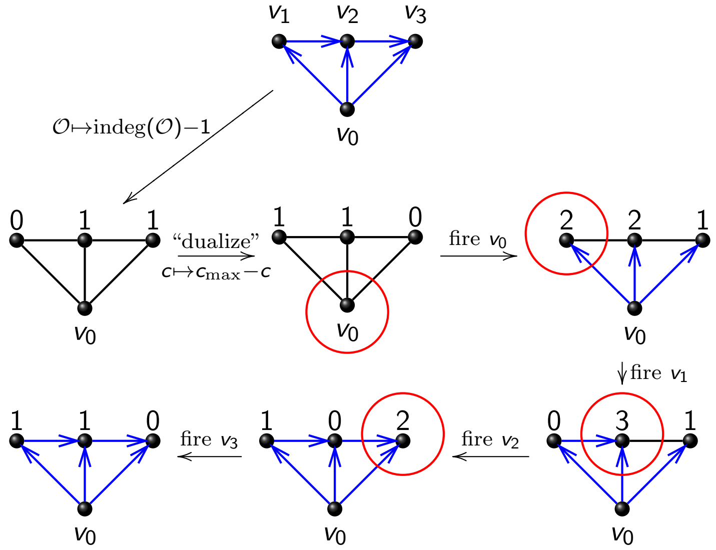
Acyclic (total) orientations and maximal parking functions

There is one class of partial orientations which is easy to see are always SHI-admissible: the acyclic (total) orientations, since they cannot have any potential cycles. It is known (Benson, Chakrabarty, Tetali) that acyclic orientations are mapped bijectively by indeg to the set of *maximal parking functions* of G° (those G°-pf that are maximal among G°-pf with the usual partial order on $\mathbb{Z}V$). The inverse of this map can be described using Dhar's algorithm, which I will discuss on the next slide.

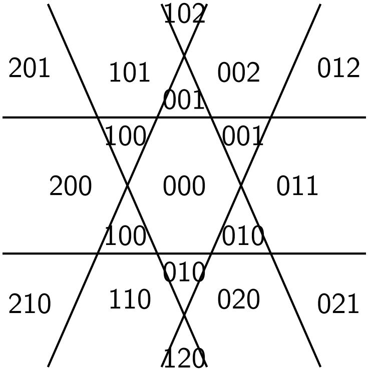
Remark

For any SHI-admissible partial orientation $\mathcal{O}$, we can find a total orientation $\mathcal{O}'$ with $\mathcal{O} \subseteq \mathcal{O}'$. This means $\mathrm{indeg}(\mathcal{O}) \leq \mathrm{indeg}(\mathcal{O}')$. It is easy to see that if c' is a G°-pf and $0 \leq c \leq c'$, then c is as well. So $\mathrm{indeg}(\mathcal{O})$ is a G°-pf. The hard part is to show that ***all*** *G°-pfs are labels of $\mathrm{SHI}(G)$.*

Dhar's algorithm and acyclic (total) orientations



The G-semiorder arrangement



We defined the G-semiorder arrangement to be the set of $2|E|$ hyperplanes

$$\mathrm{SEMI}(G) := \{x_i - x_j = 1\colon \{v_i, v_j\} \in E\}.$$

It is more symmetric than $\mathrm{SHI}(G)$: for instance, it doesn't depend on the labeling of the vertices.

SEMI-admissible partial orientations

Definition

Say $\mathcal{O}$ is SEMI-admissible if it appears as a label of $\mathrm{SEMI}(G)$.

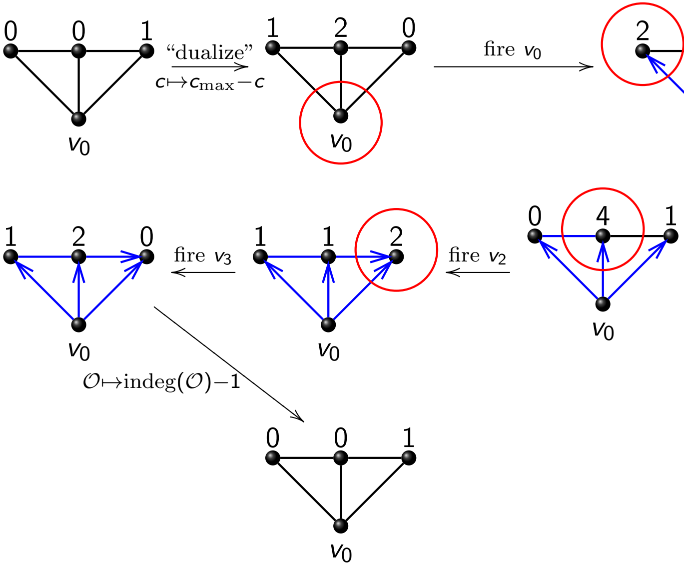
SEMI-admissible partial orientations have a much simpler description than do SHI-admissible partial orientations!

Theorem

$\mathcal{O}$ is SEMI-admissible iff every potential cycle of $\mathcal{O}$ has more blank edges than oriented edges.

We show how to extend the Dhar's algorithm map from maximal G°-pfs to acyclic total orientations of G to a map from arbitrary G°-pfs to partial orientations of G, and it turns out that the image of this map lies inside the set of SEMI-admissible orientations (precisely because of this "more blanks than arrows" condition).

Dhar's algorithm and SEMI-admissible partial orientations



v_2 was already unstable, so we mark $\{v_1, v_2\}$ blank!

DKM is true for the G-semiorder arrangement

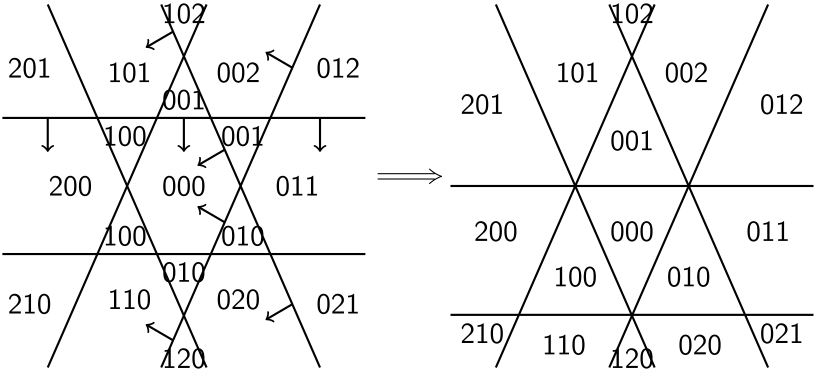
Recall that it was easy to show the set of Pak-Stanley labels of our arrangement must be a subset of the G°-parking functions. On the other hand, our Dhar's algorithm procedure shows that for any G°-parking function c, we can produce a SEMI-admissible $\mathcal{O}$ with $\mathrm{indeg}(\mathcal{O}) = c$. Thus,

Theorem

$\{$Pak-Stanley labels of $\mathrm{SEMI}(G)\} = \{$parking functions of $G^\circ\}$.

By making the problem more symmetric, we were able to solve it. But we really cared about G-Shi arrangement, not the G-semiorder arrangement! So we started playing around with what happens when we deform the G-semiorder arrangement into the G-Shi arrangement by translating the hyperplanes along their normals.

The sliding conjecture



Conjecture

As hyperplanes in $\mathrm{SEMI}(G)$ *are slid along their normals, so long as the central region is preserved, the set of Pak-Stanley labels stays the same.*

Bigraphical arrangement

Definition

For each $\{v_i, v_j\} \in E$, choose real numbers a_{ij} and a_{ji} so that there exists a $x \in \mathbb{R}^n$ with $x_i - x_j < a_{ij}$ and $x_j - x_i < a_{ji}$ for all i, j. We call $A := \{a_{ij}\}$ a *parameter list*. The *bigraphical arrangement* $\Sigma_G(A)$ is the set of $2|E|$ hyperplanes

$$\Sigma_G(A) := \{x_i - x_j = a_{ij} \colon \{v_i, v_j\} \in E\}.$$

Note: any $\Sigma_G(A)$ is isometric by a translation to a $\Sigma_G(A')$ with all $a'_{ij} > 0$. Examples of bigraphical arrangments include:

- the G-Shi arrangement (and thus Shi arrangement for $G = K_n$);
- the G-semiorder arrangement;
- interval order arrangements.

A-admissible partial orientations

Definition

For any partial orientation $\mathcal{O}$ and potential cycle C of $\mathcal{O}$, we give C a score $\nu_A(C)$:

$$\nu_A(C) := \sum_{(v_i,v_j)\in C \text{ blank in } \mathcal{O}} a_{ij} - \sum_{(v_i,v_j)\in C \text{ oriented in } \mathcal{O}} a_{ji}.$$

We say $\mathcal{O}$ is A-admissible if it appears as the label of some region of $\Sigma_G(A)$.

Theorem

$\mathcal{O}$ is A-admissible iff every potential cycle of $\mathcal{O}$ has positive score.

The "sliding" conjecture is true (so DKM is true)

Theorem

For any A and G°-parking function c, there exists an A-admissible partial orientation $\mathcal{O}$ with $\mathrm{indeg}(\mathcal{O}) = c$.

Corollary

For any parameter list A,

$$\{\textit{Pak-Stanley labels of } \Sigma_G(A)\} = \{\textit{parking functions of } G^\circ\}.$$

But we don't use anything like sliding to prove this!

Proof of “sliding” conjecture

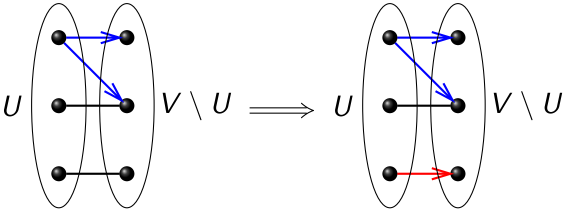
Pf: Inductive, but not really constructive. Rely on this ‘topological’ lemma.

Lemma

Let $\mathcal{O}$ be an A-admissible partial orientation and let $U \subseteq V$. Let E_U denote the edges between U and $V \setminus U$. Suppose U satisfies:

- *every edge in E_U is either oriented out of U or is blank;*
- *there is some blank edge in E_U.*

Then there is a blank edge $\{u, v\} \in E_U$ with $u \in U$ such that $\mathcal{O} \cup (u, v)$ remains A-admissible.



Enumerative properties of bigraphical arrangements

Theorem

The characteristic polynomial of a generic bigraphical arrangement is

$$\chi(\Sigma_G(\mathrm{GEN}); t) = (-2)^{n-\kappa} t^{\kappa} T_G(1 - t/2, 1).$$

Pf: Straightforward. See Stanley's notes on hyperplane arrangements.

Corollary

The number of regions and number of bounded regions of a generic bigraphical arrangement are

$$r(\Sigma_G(\mathrm{GEN})) = 2^{n-\kappa} T_G(3/2, 1)$$
$$b(\Sigma_G(\mathrm{GEN})) = 2^{n-\kappa} T_G(1/2, 1)$$

Pf: Zaslavsky's theorem.

A joke

Corollary

Suppose G is planar and G^* is its dual graph. Then the following two probabilities are equal:

- *the probability that a random partial orientation (meaning we orient each edge $\{u, v\} \in E$ as (u, v), (v, u) or blank with prob. $1/3$) of G is GEN-admissible;*
- *the probability that, after removing each edge from G^* with prob. $2/3$, the resulting graph is connected.*

Pf: follows from various interpretations of the Tutte polynomial. Would love to have a direct (multijective?) proof of this result!

Further directions

- The maximum of $r(\Sigma_G(A))$ over all A is $2^{n-\kappa}T_G(3/2,1)$. It is achieved exactly when A is generic. What is the minimum of $r(\Sigma_G(A))$ over all A? When is it achieved?
- Can we generalize bigraphical arrangements to higher dimensions (i.e., simplicial complices)? This was originally DKM's motivation.

Thank you!

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