Promotion of Kreweras words

CMS Winter Meeting, Session on Enumerative Combinatorics

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December 6th, 2020

This talk is based on joint work with Martin Rubey

Kreweras words

Recall that a *Dyck word* is a word with n A's and n B's such that every prefix has at least as many A's as B's:

AABABBAB

Dyck words of length $2n$ are of course counted by the *Catalan numbers*:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

A *Kreweras word* is a word with n A's, n B's, and n C's such that every prefix has at least as many A's as B's and at least as many as A's as C's:

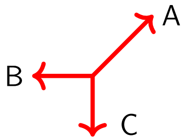
AABBCACCB

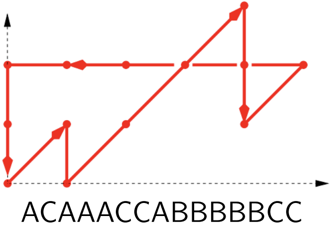
Kreweras, 1965 showed that the number of Kreweras words of length $3n$ is

$$K_n = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n} \qquad (\texttt{http://oeis.org/A006335})$$

Kreweras walks

Kreweras words are in bijection with *walks* in $\mathbb{Z}^2$ from the origin to itself with steps of the form $(1,1)$, $(-1,0)$, and $(0,-1)$, which *always remain in the nonnegative quadrant*:





These *Kreweras walks* are a fundamental example of *"walks with small steps in the quarter plane"* (see Bousquet-Mélou–Mishna, 2010).

Promotion of Kreweras words

We consider a *cyclic group action* on Kreweras words, called *promotion*.

To compute the promotion, $\mathrm{Pro}(w)$, of a Kreweras word w, find the first prefix where it has as many B's as A's or as many C's as A's and circle it:

$$w = \mathsf{AAB}\textcircled{\mathsf{B}}\mathsf{CACCB}$$

Then move the initial A to that circled position, and bump the letter there to the end of the word:

$$w = {\color{blue}\mathsf{A}}\mathsf{AB}\textcircled{{\color{purple}\mathsf{B}}}\mathsf{CACCB}$$

$$\mathrm{Pro}(w) = \mathsf{AB}{\color{blue}\mathsf{A}}\mathsf{CACCB}{\color{purple}\mathsf{B}}$$

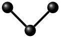
Easy to see result is still a Kreweras word, and this process is reversible.

Example of Kreweras word promotion orbit

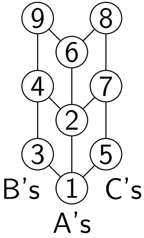
$$
\begin{aligned}
w &= \mathsf{AAB\textcircled{B}CACCB} \\
\mathrm{Pro}(w) &= \mathsf{A\textcircled{B}ACACCBB} \\
\mathrm{Pro}^2(w) &= \mathsf{AACAC\textcircled{C}BBB} \\
\mathrm{Pro}^3(w) &= \mathsf{A\textcircled{C}ACABBBC} \\
\mathrm{Pro}^4(w) &= \mathsf{AACABB\textcircled{B}CC} \\
\mathrm{Pro}^5(w) &= \mathsf{A\textcircled{C}ABBACCB} \\
\mathrm{Pro}^6(w) &= \mathsf{AAB\textcircled{B}ACCBC} \\
\mathrm{Pro}^7(w) &= \mathsf{A\textcircled{B}AACCBCB} \\
\mathrm{Pro}^8(w) &= \mathsf{AAACCB\textcircled{C}BB} \\
\mathrm{Pro}^9(w) &= \mathsf{AACCBABBC}
\end{aligned}
$$

Observe: $\mathrm{Pro}^{3n}(w) =$ swap B's and C's in w; so $\mathrm{Pro}^{6n}(w) = w$.

Kreweras words as linear extensions

Let $V(n) :=$  $\times [n]$ be the product of the 3-element "V"-shaped poset and the n-element chain $[n]$. Kreweras–Niederhausen, 1981 observed that Kreweras words of length $3n$ correspond to *linear extensions* of $V(n)$:

AABBCACCB $\Leftrightarrow$



Compare: Dyck words of length $2n$ correspond to linear extensions of the poset $[2]x[n]$, i.e., $2 \times n$ *Standard Young Tableaux (SYTs)*:

AABABBAB $\Leftrightarrow$

1	2	4	7
3	5	6	8

Schützenberger's promotion

Promotion is the following invertible operation on SYTs of shape λ:

- Delete the entry 1.
- Slide boxes into the resulting hole.
- Decrement all entries.
- Fill the hole with n.

Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{del. 1}} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline \bullet & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline 2 & \bullet & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & \bullet & 6 \\ \hline \end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline 2 & 4 & 5 \\ \hline 3 & 6 & \bullet \\ \hline \end{array} \xrightarrow{\text{d}}$$

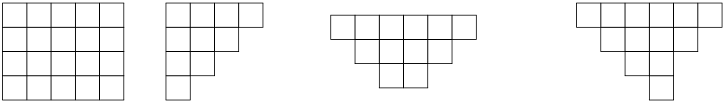
Together with related operation *evacuation*, promotion was defined by Schützenberger to study the *RSK algorithm*.

There is a straightforward extension of promotion to linear extensions of any poset, and case of $V(n) =$ promotion of Kreweras words.

Shapes with good promotion behavior

Promotion behaves chaotically for most shapes, but:

- (Schützenberger, 1977) For P a *rectangle*, $\mathrm{Pro}^{\#P}$ is the identity.
- (Edelman–Greene, 1987) For P a *staircase*, $\mathrm{Pro}^{\#P}$ is *transposition*.
- (Haiman, 1992) For P a *shifted trapezoid* or *shifted double staircase*, $\mathrm{Pro}^{\#P}$ is the identity.
- (Haiman–Kim, 1992) These are the **only** four families of shapes with good promotion behavior.



Rectangle   Staircase   Shifted trapezoid   Shifted double staircase

Stanley's Question (2009): Any other posets P with good Pro behavior?

$V(n)$ has good promotion behavior

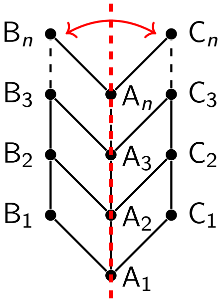
Example

$$w = \text{AAB}\textcircled{\text{B}}\text{CACCB}$$
$$\mathrm{Pro}(w) = \text{A}\textcircled{\text{B}}\text{ACACCBB}$$
$$\mathrm{Pro}^2(w) = \text{AACAC}\textcircled{\text{C}}\text{BBB}$$
$$\mathrm{Pro}^3(w) = \text{A}\textcircled{\text{C}}\text{ACABBBC}$$
$$\mathrm{Pro}^4(w) = \text{AACABB}\textcircled{\text{B}}\text{CC}$$
$$\mathrm{Pro}^5(w) = \text{A}\textcircled{\text{C}}\text{ABBACCB}$$
$$\mathrm{Pro}^6(w) = \text{AAB}\textcircled{\text{B}}\text{ACCBC}$$
$$\mathrm{Pro}^7(w) = \text{A}\textcircled{\text{B}}\text{AACCBCB}$$
$$\mathrm{Pro}^8(w) = \text{AAACCB}\textcircled{\text{C}}\text{BB}$$
$$\mathrm{Pro}^9(w) = \text{AACCBABBC}$$

We addressed Stanley's question:

Theorem (H.-Rubey, 2020)

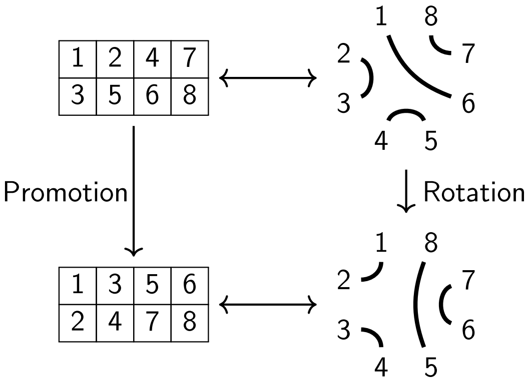
For $P = V(n)$, $\mathrm{Pro}^{\#P}$ is reflection across the vertical axis of symmetry.



Promotion = rotation: noncrossing matchings

How to prove good behavior of promotion? One way: find a *diagrammatic model* of the linear extensions where *promotion = rotation*.

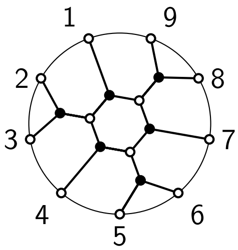
Dennis White observed that under a natural bijection between $2 \times n$ SYTs and noncrossing matchings of $[2n]$, promotion corresponds to rotation:



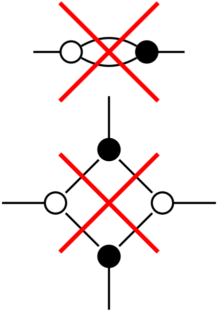
Webs

Webs are certain planar graphs introduced by Kuperberg, 1996 to study the invariant theory of simple Lie algebras and their quantum groups.

An $\mathfrak{sl}_3$-*web* is a bipartite planar graph drawn inside a disk, with boundary vertices of degree 1 and all internal vertices of degree 3:



An $\mathfrak{sl}_3$-web is *irreducible* if all internal faces have ≥ 6 sides:



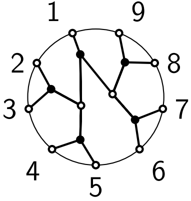
Promotion = rotation: webs

Khovanov–Kuperberg, 1999 described a bijection between $3 \times n$ SYTs and irreducible $\mathfrak{sl}_3$-webs with $3n$ white boundary vertices.

Petersen–Pylyavskyy–Rhoades, 2009 (see also Tymoczko, 2012) showed that, again, promotion is rotation:

1	2	6
3	4	8
5	7	9

⟷

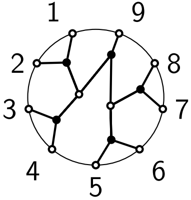


Promotion ↓

↓ Rotation

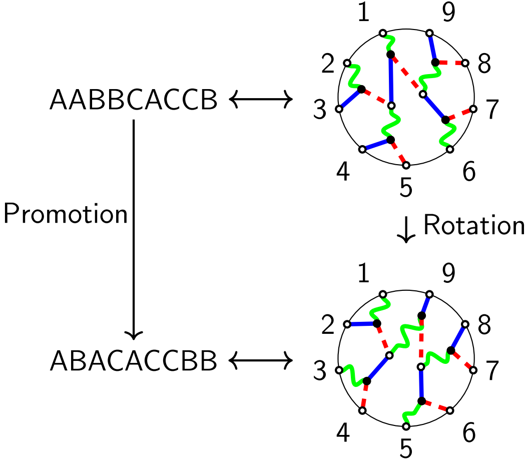
1	3	5
2	6	7
4	8	9

⟷



Kreweras words as edge-colored webs

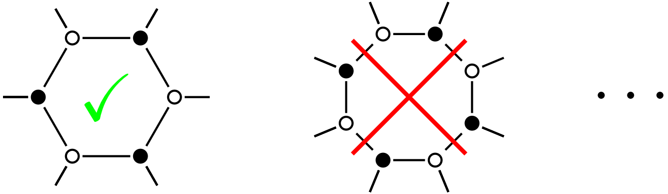
We showed that Kreweras words can be encoded as certain *3-edge-colored* $\mathfrak{sl}_3$-webs so that once again promotion corresponds to rotation:



(**Note**: the web rotates, but the edge-coloring does slightly more.)

Kreweras webs

A *Kreweras web* is an irreducible $\mathfrak{sl}_3$-web with $3n$ white boundary vertices such that **no internal faces have a multiple of 4 sides**:



Theorem (H.-Rubey, 2020)

- *The webs that come from Kreweras words are Kreweras webs.*
- *For each Kreweras web $\mathcal{W}$, there are $2^{\kappa(\mathcal{W})}$ ways to 3-edge-color it so that it corresponds to a Kreweras word, where $\kappa(\mathcal{W})$ is the number of connected components of $\mathcal{W}$.*

Enumerative corollaries on webs

Corollary (H.-Rubey, 2020)

We have

$$\sum_{\mathcal{W}} 2^{\kappa(\mathcal{W})} = K_n = \frac{4^n}{(n+1)(2n+1)}\binom{3n}{n},$$

where the sum is over all Kreweras webs $\mathcal{W}$.

Moreover, the number of connected Kreweras webs is

$$2^n \frac{(4n-3)!}{(3n-1)!n!} \qquad \text{(c.f. http://oeis.org/A000260).}$$

Open problems

- Irreducible $\mathfrak{sl}_3$ webs with all white boundary vertices give a basis of space of invariant tensors, in fact, its symmetric group representation. What is the **algebraic** significance of the Kreweras webs?
- Related to last point: *cyclic sieving* for promotion of Kreweras words.
- Bernardi, 2007 bijectively related Kreweras words and (rooted, bridgeless) cubic maps. We bijectively related Kreweras words and Kreweras webs. Cubic maps and webs feel very similar. Is there a **direct** link between cubic maps and webs?

Thank you!

- These slides are on my website.
- The paper is at: `arXiv:2005.14031`