Parking functions and tree inversions

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based on joint work with Petar Gaydarov

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Parking functions

A **parking function** of length n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive integers whose weakly increasing rearrangement $\alpha_{i_1} \leq \alpha_{i_2} \leq \cdots \leq \alpha_{i_n}$ satisfies $\alpha_{i_j} \leq j$ for all $j = 1, \ldots, n$.

Let $\mathrm{PF}(n) = \{\text{parking functions of length } n\}$. For example:

$$\mathrm{PF}(2) = \{(1,1),\ (1,2),\ (2,1)\}$$

Their name comes from an interpretation in terms of parking cars.

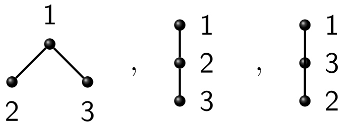
Theorem (Konheim and Weiss 1966, Pollak 1974)

$\#\mathrm{PF}(n) = (n+1)^{n-1}$

This same sequence of numbers appears in several other contexts...

Labeled trees

Let $\mathrm{Tree}(n) = \{\text{labeled trees on vertex set } \{1, 2, \ldots, n\}\}$. For example:

$\mathrm{Tree}(3) = \Bigg\{$ 

$\Bigg\}$

Theorem (Borchardt 1860, Cayley 1889)

$\#\mathrm{Tree}(n+1) = (n+1)^{n-1}$

There are many known bijections between parking functions and trees, and these two classes of combinatorial objects have more connections too...

Inversions in permutations

Let S_n be the symmetric group of permutations of $\{1, \ldots, n\}$.

Recall that for a permutation $w = w_1 \ldots w_n \in S_n$ (in one-line notation), an **inversion** of w is a pair (i, j) of indices $1 \leq i < j \leq n$ with $w_i > w_j$.

We use $\mathrm{inv}(w)$ to denote the number of inversions of w. For example:

$$w = 1532476 \ \Rightarrow \ \mathrm{inv}(w) = \#\{(2,3),(2,4),(2,5),(3,4),(6,7)\} = 5$$

Theorem (Rodrigues 1839)

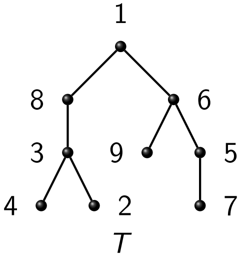
$$\sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q!,$$

where we use standard "q-notation" $[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1}$ *and* $[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q$.

Inversions in trees

Let $T \in \mathrm{Tree}(n)$. We write $i \leq_T j$ to mean i appears in the unique path from j to 1. An **inversion** of T is a pair (i, j) of vertices $1 \leq i < j \leq n$ with $j \leq_T i$. We use $\mathrm{inv}(T)$ to denote the number of inversion of T.

For example:



$$\Rightarrow \quad \mathrm{inv}(T) = \#\{(2,3),(2,8),(3,8),(4,8),(5,6)\}$$
$$= 5$$

Tree inversions generalize permutation inversions for “linear” trees.

Co-sum of parking functions & Kreweras's result

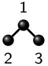
For $\alpha \in \mathrm{PF}(n)$, we define its **co-sum** to be $\mathrm{cosum}(\alpha) = \binom{n+1}{2} - \sum_{i=1}^{n} \alpha_i$.
Parking functions of maximal sum have co-sum zero.

Theorem (Kreweras 1980)

$\sum_{\alpha \in \mathrm{PF}(n)} q^{\mathrm{cosum}(\alpha)} = \sum_{T \in \mathrm{Tree}(n+1)} q^{\mathrm{inv}(T)}$

For example, for $n = 2$:

$q^{\mathrm{cosum}(1,1)} + q^{\mathrm{cosum}(1,2)} + q^{\mathrm{cosum}(2,1)} = q + 2 = q^{\mathrm{inv}(T_1)} + q^{\mathrm{inv}(T_2)} + q^{\mathrm{inv}(T_3)}$



Kreweras's original proof was via generating functions. Bijective proofs were later given by Shin (2007), Guedes de Oliveira & Las Vergnas (2011), and Perkinson, Yang, & Yu (2017).

Vector parking functions

We now consider a variant of parking functions. Let $\mathbb{N} = \{0, 1, \ldots\}$ and let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ be a nonnegative integer vector.

An **x-parking function** is a sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive integers whose increasing rearrangement $a_{i_1} \leq \cdots \leq a_{i_n}$ satisfies $a_{i_j} \leq x_1 + \cdots + x_j$ for all $j = 1, \ldots, n$. We let $\mathrm{PF}(\mathbf{x}) = \{\mathbf{x}\text{-parking functions}\}$.

For example:

$$\mathbf{x} = (1, 2) \;\Rightarrow\; \mathrm{PF}(\mathbf{x}) = \{(1,1), (1,2), (2,1), (1,3), (3,1)\}$$

Notice that $\mathrm{PF}(n) = \mathrm{PF}(\mathbf{x})$ where $\mathbf{x} = (1, 1, \ldots, 1)$. Also note that **rational parking functions** are a special case of **x**-parking functions.

Enumerating vector parking functions

Define the set $\Gamma(n)$ of nonnegative integer vectors by

$$\Gamma(n) = \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n \colon \sum_{i=1}^{j} \gamma_i \geq j \text{ for all } 1 \leq j \leq n-1 \text{ and } \sum_{i=1}^{n} \gamma_i = n\}$$

It is well-known that $\#\Gamma(n) = C_n = \frac{1}{n+1}\binom{2n}{n}$, the **Catalan number**.

Theorem (Pitman and Stanley 2002)

For any $\mathbf{x} \in \mathbb{N}^n$,

$$\begin{aligned}\#\mathrm{PF}(\mathbf{x}) &= \sum_{\alpha \in \mathrm{PF}(n)} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n} \\ &= \sum_{\gamma \in \Gamma(n)} \frac{n!}{\gamma_1! \gamma_2! \cdots \gamma_n!} x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}\end{aligned}$$

Co-sum enumerator of vector parking functions

Kung and Yan (2003) noted that Pitman–Stanley's work could also be used to obtain the co-sum enumerator of vector parking functions:

Theorem (Kung and Yan 2003)

For any $\mathbf{x} \in \mathbb{N}^n$,

$$\sum_{\alpha \in \mathrm{PF}(\mathbf{x})} q^{\mathrm{cosum}(\alpha)} = \sum_{\gamma \in \Gamma(n)} \frac{n!}{\gamma_1! \gamma_2! \cdots \gamma_n!} q^{\sum_{i=1}^{n} (\gamma_1 + \gamma_2 + \cdots + \gamma_i - i) x_{i+1}} \prod_{i=1}^{n} [x_i]_q^{\gamma_i}$$

This is a useful formula (a sum over Catalan many terms), but it does not obviously reduce to Kreweras's result in the case when $\mathbf{x} = (1, 1, \ldots, 1)$.

Is there some formula involving trees? In fact, there is...

Rooted plane trees

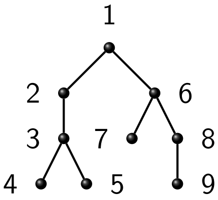
A **rooted plane tree** is a $T \in \mathrm{Tree}(n)$ for which:

- $i \leq_T j$ implies $i \leq j$ for all $1 \leq i, j \leq n$;
- $i \leq_T k$ implies $i \leq_T j$ for all $1 \leq i < j < k \leq n$.

(Can just think of this as saying the tree is labeled in **depth-first order**.)

Let $\mathrm{RPT}(n) = \{\text{rooted plane trees in } \mathrm{Tree}(n)\}$.

For example, an element of $\mathrm{RPT}(9)$ is:



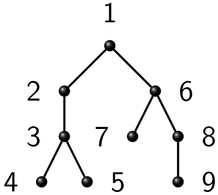
It is well-known that $\#\mathrm{RPT}(n+1) = C_n$, again the Catalan number.

Admissible vertex orders and inversions

Let $T \in \mathrm{RPT}(n)$. An **admissible vertex order** of T is a total order $\prec$ on the non-root vertices (i.e., $\{2,\ldots,n\}$) for which $i < j$ with i and j **siblings** in T implies that $j \prec i$. Let $\mathrm{AVO}(T) = \{\text{admissible vertex orders of } T\}$.

For $\prec \in \mathrm{AVO}(T)$, an **inversion** of $\prec$ is a pair (i,j) with $i \leq_T j$ but $j \prec i$.

For example:



T

$5 \prec 3 \prec 4 \prec 8 \prec 6 \prec 9 \prec 2 \prec 7$

$\Rightarrow$ inversions of T w.r.t. $\prec$ $= \{(2,3),(2,4),(2,5),(3,5),(6,8)\}$

Co-sum enumerator of vector parking functions, again

Theorem (Gaydarov and Hopkins 2015)

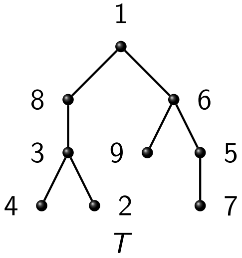
For any $\mathbf{x} \in \mathbb{N}^n$,

$$\sum_{\alpha \in \mathrm{PF}(\mathbf{x})} q^{\mathrm{cosum}(\alpha)} = \sum_{T \in \mathrm{RPT}(n+1)} \left(\sum_{\prec \in \mathrm{AVO}(T)} q^{\sum_{\substack{i \leq_T j, \\ j \prec i}} x_{\mathrm{par}_T(i)}} \right) \prod_{i=1}^{n} [x_i]_q^{\#\mathrm{child}_T(i)}$$

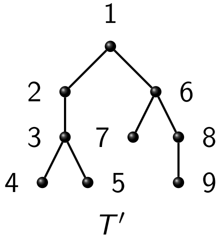
where $\mathrm{par}_T(i)$ *denotes the* **parent** *of the vertex* i *in* T, *and* $\#\mathrm{child}_T(i)$ *denotes the number of* **children** *of the vertex* i *in* T.

This reduces to Kreweras's result when $\mathbf{x} = (1, 1, \ldots, 1)$, because...

Given any $T \in \mathrm{Tree}(n)$, we can produce a pair $(T', \prec)$ with $T' \in \mathrm{RPT}(n)$ and $\prec \in \mathrm{AVO}(T)$ by performing a **depth-first search** of T, starting at the root 1, and always preferring to visit the vertex with the largest label:



$\Rightarrow$



$5 \prec 3 \prec 4 \prec 8 \prec 6 \prec 9 \prec 2 \prec 7$

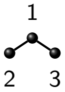
This procedure gives a bijective correspondence, and it preserves inversions.

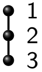
Comparing the formulas, $n = 2$

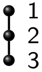
$\Gamma(2) = \{(2,0),(1,1)\}$ so the Kung and Yan formulas says:

$$\sum_{\alpha \in \mathrm{PF}(x_1,x_2)} q^{\mathrm{cosum}(\alpha)} = q^{x_2}[x_1]^2 + 2[x_1][x_2].$$

The pairs $(T, \prec)$ with $T \in \mathrm{RPT}(3)$ and $\prec \in \mathrm{AVO}(T)$ are

(

$, 3 \prec 2$) , (

$, 2 \prec 3$) , (

$, 3 \prec 2$)

so the Gaydarov and Hopkins formula says:

$$\sum_{\alpha \in \mathrm{PF}(x_1,x_2)} q^{\mathrm{cosum}(\alpha)} = [x_1]^2 + (1 + q^{x_1})[x_1][x_2].$$

In fact, $q^{x_2}[x_1]^2 + 2[x_1][x_2] = [x_1]^2 + (1 + q^{x_1})[x_1][x_2]$ for all $(x_1, x_2) \in \mathbb{N}^2$.

Comparing the formulas, $n = 3$

Comparing the two formulas for $n = 3$ implies that

$$q^{2x_2+x_3}[x_1]_q^3 + 3q^{x_2+x_3}[x_1]_q^2[x_2]_q + 3q^{x_2}[x_1]_q^2[x_3]_q + 3q^{x_3}[x_1]_q[x_2]_q^2 + 6[x_1]_q[x_2]_q[x_3]_q$$
$$= [x_1]_q^3 + (1+2q^{x_1})[x_1]_q^2[x_2]_q + (2+q^{x_1})[x_1]_q^2[x_3]_q + (1+q^{x_1}+q^{2x_1})[x_1]_q[x_2]_q^2$$
$$+(1+q^{x_1}+q^{x_2}+q^{2x_1}+q^{x_1+x_2}+q^{2x_1+x_2})[x_1]_q[x_2]_q[x_3]_q$$

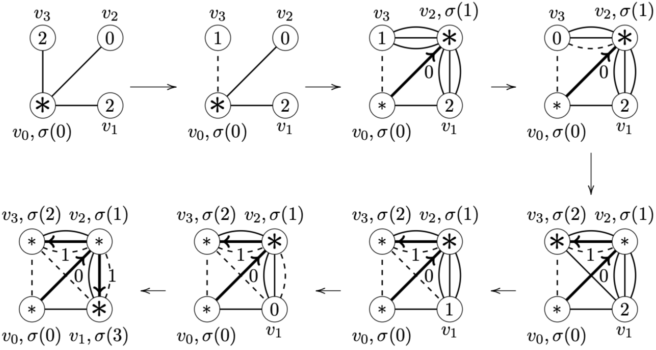
for all $(x_1, x_2, x_3) \in \mathbb{N}^3$.

This identity is true but far from obvious!

Notice how the "monomials" $[x_1]_q^{c_1}[x_2]_q^{c_2}\cdots[x_n]_q^{c_n}$ on both sides are the same, but the "coefficients" in front of these are different.

The proof: chip-firing

The proof of our formula uses ideas from **chip-firing**, specifically, a version of **Dhar's burning algorithm** due to Perkinson, Yang, and Yu (2017).



However, note that **graphical parking functions** are *different* from vector parking functions, so we had to use a "symmetrization" trick.

Conclusion

I end with a couple of scattershot thoughts related to this research.

- Even if you already have one formula, maybe you can find another!
- With your favorite variant of parking functions, it might be interesting to study the (co-)sum enumerator of these parking functions, and try to find a connection to trees and their inversions!
- This research was done in 2014 at RSI, a summer program run by MIT for talented high school students to engage in scientific research. My mentee Petar Gaydarov was a high school student from Bulgaria. I'm very proud of the work he did; he won a "Karl Menger Memorial Prize" for this work. All of us, in America and throughout the world, benefit tremendously from scientific exchange between countries!

Thank you!

see `arXiv:1506.03470` for the paper