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Invariance of rowmotion for variants of the Tamari lattice

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Abstract. We show that the rowmotion operator from dynamical algebraic combinatorics behaves the same on all alt ν-Tamari lattices for a fixed path ν. We use this invariance of rowmotion to establish cyclic sieving and homomesy results for rational Tamari lattices. We also conjecture that this invariance extends to the more general cross Tamari lattices.

Keywords: rowmotion, alt ν- and cross Tamari lattices, cyclic sieving, homomesy

1 Introduction

This is an extended abstract only; for the full version of our paper, with proofs, see [1].

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Rowmotion is an invertible operator that was originally defined on the set $J(P)$ of order ideals of a finite poset P [11]. By viewing $J(P)$ as a finite distributive lattice, the definition of rowmotion has more recently been extended to act on broader classes of finite lattices, such as semidistributive lattices, trim lattices, and beyond [3, 12].

Rowmotion is one of the operators that has been most intensively studied from the perspective of dynamical algebraic combinatorics. Two major themes in this field concern the orbit structures of operators acting on combinatorial sets (e.g., the *cyclic sieving phenomenon*) and the behavior of statistics along orbits of these operators (e.g., *homomesy*).

It is known [2, 3] that rowmotion behaves "the same," with respect to orbit structures and statistical orbit averages, on the (distributive) *Stanley lattice* of Dyck paths of length $2n$ ordered by nesting and on the (semidistributive) *Tamari lattice* of triangulations of an $(n+2)$-gon. We dramatically generalize this example. Specifically, we study rowmotion on the *alt ν-Tamari lattices* [6]. Our main result is:[1]

Theorem 1. *Fix a path ν, and let L_1 and L_2 be two alt ν-Tamari lattices. Then rowmotion behaves the same for L_1 and L_2. That is, there is a bijection $\Phi\colon L_1 \to L_2$ such that:*

- $\Phi(\mathrm{Row}(x)) = \mathrm{Row}(\Phi(x))$ *for all* $x \in L_1$;
- $\sum_{y \in \{\mathrm{Row}^j(x)\colon j \geq 0\}} \mathrm{ddeg}(y) = \sum_{y \in \{\mathrm{Row}^j(x)\colon j \geq 0\}} \mathrm{ddeg}(\Phi(y))$ *for all* $x \in L_1$,

where Row *is rowmotion and* ddeg *the* down-degree *(i.e., number of lower covers) statistic.*

In the case where ν is a staircase path, Theorem 1 recovers the invariance of rowmotion for the Stanley and Tamari lattices, and in fact interpolates between these. For example, for the staircase path $\nu = (EN)^3$, we depict all the rowmotion orbits for the four alt ν-Tamari lattices in Figure 1, with the Tamari lattice on the left and Stanley lattice on right. Note the orbit sizes of 8, 4, and 2, and down-degree orbit averages of $\frac{3}{2}$.

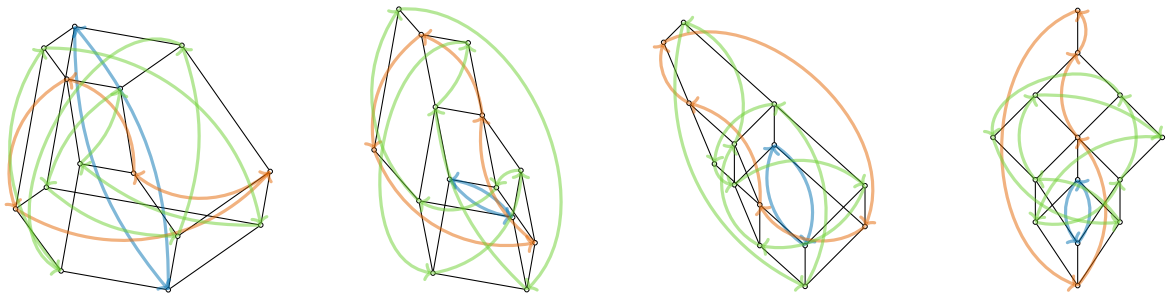

Figure 1: Rowmotion orbits on all alt ν-Tamari lattices for $\nu = ENENEN$.

We use Theorem 1 to prove cyclic sieving and homomesy results for rowmotion acting on the *rational Tamari lattices*, as conjectured in [12, 7]. We also conjecture that Theorem 1 extends to the more general *cross Tamari lattices* [4].

[1] The independent, simultaneous work of [8] also proves Theorem 1 in some special cases.

2 Lattices and rowmotion

We follow standard notation and terminology for posets and lattices. **All posets here are assumed to be finite.** Let L be a lattice. We use $\mathcal{J}_L$ (respectively, $\mathcal{M}_L$) to denote the set of *join-irreducible* (resp., *meet-irreducible*) elements of L.

We say L is *semidistributive* if for all $x, y, z \in L$, we have the following implications:

$$x \vee y = x \vee z \Longrightarrow x \vee (y \wedge z) = x \vee y$$
$$x \wedge y = x \wedge z \Longrightarrow x \wedge (y \vee z) = x \wedge y.$$

Suppose L is semidistributive, and let $x \lessdot y$ be a cover relation in L. It follows from the definition of semidistributivity that $\{z \in L : z \vee x = y\}$ has a minimum element $j_{x,y}$ and that this element is join-irreducible. We view $j_{x,y}$ as a label assigned to the edge $x \lessdot y$ in the Hasse diagram of L. Thus, we obtain a *canonical labeling* of the edges of L with the elements of $\mathcal{J}_L$. For $x \in L$, we let $\mathcal{D}_L(x) := \{j_{w,x} : w \lessdot x\}$ and $\mathcal{U}_L(x) := \{j_{x,w} : x \lessdot w\}$ be the sets of labels of edges going down from x and going up from x, respectively. It is known that the maps $\mathcal{D}_L$ and $\mathcal{U}_L$ are injective on L. Moreover, the sets $\{\mathcal{D}_L(x) : x \in L\}$ and $\{\mathcal{U}_L(x) : x \in L\}$ are equal, and they form an abstract simplicial complex known as the *canonical join complex* of L; see [3]. This implies that there is a unique bijection, called *rowmotion* and denoted $\mathrm{Row}\colon L \to L$, satisfying $\mathcal{U}_L(x) = \mathcal{D}_L(\mathrm{Row}(x))$ for all $x \in L$. Note in particular that rowmotion restricts to a bijection from $\mathcal{M}_L$ to $\mathcal{J}_L$.

The *length* of a poset P, denoted $\ell(P)$, is the maximum length of a chain in P, where the length of the chain $x_0 < x_1 < \cdots < x_k$ is k. It is straightforward to show that if L is a lattice, then $\ell(L) \leq \min\{|\mathcal{J}_L|, |\mathcal{M}_L|\}$. We say that L is *extremal* if $\ell(L) = |\mathcal{J}_L| = |\mathcal{M}_L|$. Note that all distributive lattices are both semidistributive and extremal. In fact, the lattices we work with here will all be semidistributive and extremal.

When the lattice L is both semidistributive and extremal, there is an alternative method for computing rowmotion "in slow motion" due to Thomas and Williams [12], which we now explain. Let L be a semidistributive lattice. For each join-irreducible element $j \in \mathcal{J}_L$ and each element $x \in L$, let

$$\tau_j(x) := \begin{cases} y & \text{if } j_{x,y} = j \in \mathcal{U}_L(x) \text{ or } j_{y,x} = j \in \mathcal{D}_L(x); \\ x & \text{otherwise.} \end{cases}$$

In other words, to obtain $\tau_j(x)$ from x, we look for an edge in the Hasse diagram of L incident to x and labeled by j. If such an edge exists, then it is unique and we walk along it; if no such edge exists, we stay at x. This defines for each $j \in \mathcal{J}_L$ an involution $\tau_j\colon L \to L$ called a *toggle*.

Theorem 2 (Thomas and Williams [12])**.** *Let L be a semidistributive and extremal lattice, and let $z_{N+1} \lessdot z_N \lessdot \cdots z_2 \lessdot z_1$ be a maximum-length chain of L, where $N = \ell(L) = |\mathcal{J}_L| = |\mathcal{M}_L|$. Then $\mathrm{Row} = \tau_{j_N} \circ \cdots \circ \tau_{j_1}$, where $j_i := j_{z_{i+1},z_i}$ for $i = 1, \ldots, N$.*

Example 1. *Consider a distributive lattice $L = J(P)$ of order ideals of a poset P. For each $p \in P$, let $\Lambda(p) = \{q \in P : q \leq p\}$ be the order ideal generated by p. The map $p \mapsto \Lambda(p)$ is a bijection from P to $\mathcal{J}_L$. If $u \lessdot v$ in L, there is a unique $r \in P$ with $v = u \sqcup \{r\}$, and the join-irreducible element $j_{u,v}$ is $\Lambda(r)$. Thus, the toggles $\tau_p := \tau_{\Lambda(p)} : L \to L$ for $p \in P$ are given by*

$$\tau_p(I) = \begin{cases} I \triangle \{p\} & \text{if } I \triangle \{p\} \in J(P); \\ I & \text{otherwise,} \end{cases}$$

where $\triangle$ denotes symmetric difference. And $\mathrm{Row} = \tau_{p_N} \circ \cdots \circ \tau_{p_1}$, where $p_N, \ldots, p_1$ is any linear extension of P. This "row-by-row" way to compute rowmotion explains the name [11].

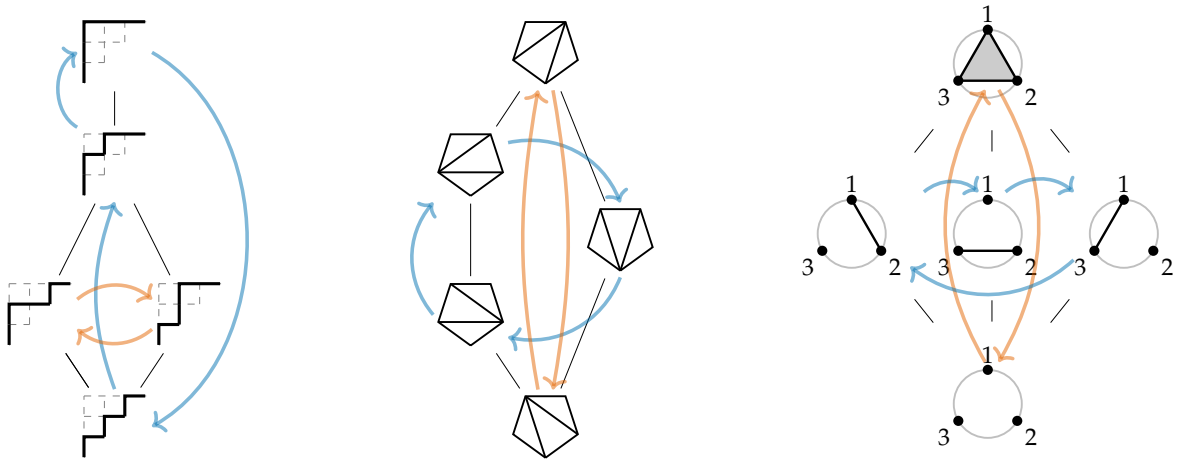

Figure 2: The Stanley (left), Tamari (middle), and noncrossing partition (right) lattices, with rowmotion (left and middle) and Kreweras complement (right) orbits, for $n = 3$.

Example 2. *Let $n \geq 1$ and let $\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number. The Stanley lattice, $\mathrm{Stan}(n)$, is the poset of Dyck paths of length $2n$ ordered by nesting, a distributive lattice on $\mathrm{Cat}(n)$-many elements. Meanwhile, the Tamari lattice, $\mathrm{Tam}(n)$, is another semidistributive and extremal lattice on $\mathrm{Cat}(n)$-many elements, whose cover relations can be described in terms of flips in triangulations or rotations of trees. Recall from Section 1 that $\mathrm{Stan}(n)$ and $\mathrm{Tam}(n)$ have the same rowmotion behavior. One way to relate rowmotion on $\mathrm{Stan}(n)$ and $\mathrm{Tam}(n)$ is via another operation on a third Catalan lattice. The noncrossing partition lattice is the collection of set partitions of $\{1, 2, \ldots, n\}$ whose blocks are noncrossing when the numbers $1, 2, \ldots, n$ are arranged clockwise around a circle, partially ordered by refinement. The Kreweras complement is a complementation map on the noncrossing partition lattice, whose square is rotation. It is known that both rowmotion on the Stanley lattice [2], and rowmotion on the Tamari lattice [3], have the same orbit structure as Kreweras complement of noncrossing partitions. This is depicted, in the case $n = 3$, in Figure 2.*

3 Alt ν-Tamari lattices

The alt ν-Tamari lattices are a family of finite lattices, introduced recently by Ceballos and Chenevière [6], which depend on two parameters: ν, an east-north path, and δ, a vector of nonnegative integers. These lattices generalize both the ν-Tamari lattices [10] and the distributive lattices of ν-Dyck paths, and in fact interpolate between these two cases. In [6], two definitions of the alt ν-Tamari lattices are presented: one in terms of paths, and another in terms of trees. The tree definition will be most convenient for us.

Let ν be a finite *path* in the grid $\mathbb{Z}^2$, starting at $(0,0)$, consisting of unit east and north steps. We write $\nu = E^{v_0}NE^{v_1}N\cdots NE^{v_n}$ and in this way also identify ν with the vector $\nu = (\nu_0, \nu_1, \ldots, \nu_n)$. We think of ν as tracing the southeast boundary of the *Young diagram* of the partition $\lambda = (\nu_{n-1} + \cdots + \nu_0, \nu_{n-2} + \cdots + \nu_0, \ldots, \nu_1 + \nu_0, \nu_0)$, and write F_ν for this associated Young diagram. An *increment vector*, with respect to ν, is a vector $\delta = (\delta_1, \ldots, \delta_n)$ of nonnegative integers satisfying $\delta_i \leq \nu_i$ for all i. A *stack polyomino* is a collection of boxes in $\mathbb{Z}^2$ that can be obtained from a Young diagram by permuting the columns in such a way that, when read from left to right, the column lengths first increase and then decrease. Given an increment vector δ for the path ν, we produce the stack polyomino $F_{\delta,\nu}$ by taking the Young diagram F_ν and permuting its columns so that δ_i is the number of columns of height $n-i$ to the right of the largest column(s), for all i. For example, Figure 3 depicts the Young diagram F_ν on the left and the stack polyomino $F_{\delta,\nu}$ on the right, for $\nu = (1,4,1,0,0)$ and $\delta = (2,1,0,0)$.

Fix a path ν and increment vector δ. Let p and q be two grid points in the stack polyomino $F_{\delta,\nu}$. We say that p and q are *incompatible* if p lies strictly to the southwest of q and the smallest rectangle containing p and q lies completely inside of $F_{\delta,\nu}$. A *(δ,ν)-tree* is a maximal collection of pairwise compatible grid points in $F_{\delta,\nu}$. For example, the right of Figure 3 depicts a (δ,ν)-tree T for the aforementioned ν and δ. For any increment vector δ, (δ,ν)-trees are in bijection with *ν-Dyck paths*, i.e., east-north paths ν' with the same number of E's and N's as ν and staying weakly northwest of it (equivalently, with $F_{\nu'} \subseteq F_\nu$). This bijection is *right-flushing*, as shown in Figure 3.

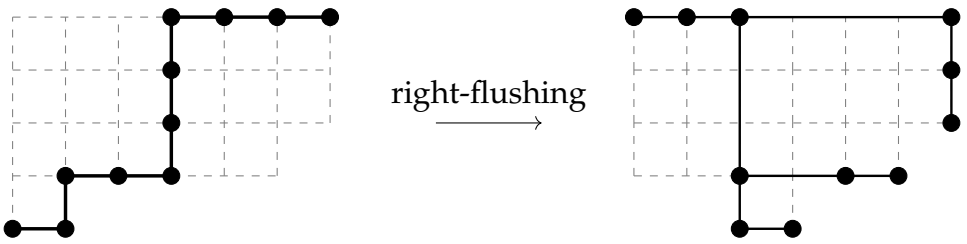

Figure 3: Right-flushing of a ν-Dyck path to a (δ,ν)-tree T for $\nu = (1,4,1,0,0)$, $\delta = (2,1,0,0)$. Both have 2, 3, 1, 1, and 4 nodes from bottom to top.

Given a (δ, ν)-tree T, we can always draw edges along grid lines connecting the nodes (i.e., points) in T in such a way that it really is a tree, rooted at the northwestern-most node. Let q be a node of T adjacent to a node p north of it and a node r east of it. Let q' be the fourth point of the smallest rectangle containing p, q, and r. We call the tree T' obtained from T by replacing q with q' a *right-rotation* of T (and T is a *left-rotation* of T'). We call the associated box $b_{T,T'}$ which has r as its southeast corner an *ascent box* of T (and a *descent box* of T'). We say the rotation happens at this box $b_{T,T'}$. See Figure 4.

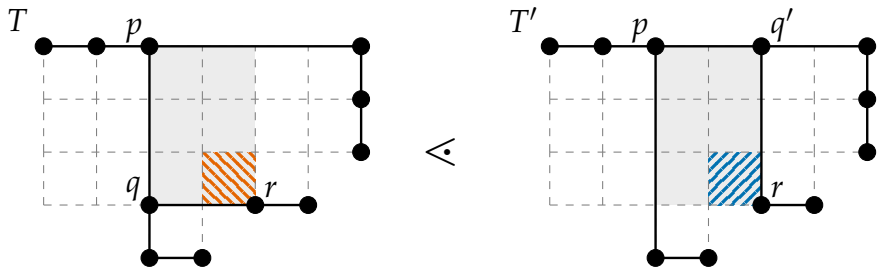

Figure 4: Rotation in a (δ, ν)-tree ($\nu = (1, 4, 1, 0, 0)$, $\delta = (2, 1, 0, 0)$) at box $b_{T,T'}$ (striped).

The *alt ν-Tamari lattice* [6] $\mathrm{Tam}_\nu(\delta)$ is the poset on the (δ, ν)-trees T whose cover relations are $T \lessdot T'$ when T' is obtained from T by a right-rotation. For example, Figure 5 shows all the alt ν-Tamari lattices for $\nu = (1, 2, 0)$. $\mathrm{Tam}_\nu(\delta)$ is always a semistributive and extremal lattice. For $\delta = (\nu_1, \ldots, \nu_n)$, $\mathrm{Tam}_\nu(\delta)$ is the *ν-Tamari lattice* [10], and for $\delta = (0, \ldots, 0)$, $\mathrm{Tam}_\nu(\delta)$ is the distributive lattice of ν-Dyck paths ordered by nesting. In particular, for $\nu = (EN)^{n-1}$, we recover the lattices $\mathrm{Tam}(n)$ and $\mathrm{Stan}(n)$ in this way.

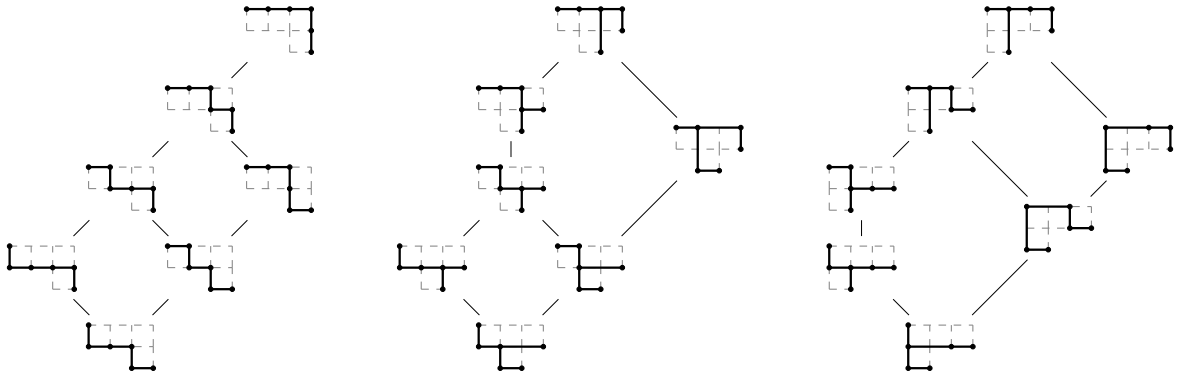

Figure 5: Examples of alt ν-Tamari lattices $\mathrm{Tam}_\nu(\delta)$ for $\nu = ENEEN = (1, 2, 0)$. Left: the ν-Dyck lattice, $\delta = (0, 0)$. Middle: $\delta = (1, 0)$. Right: the ν-Tamari lattice, $\delta = (2, 0)$.

Since $\mathrm{Tam}_\nu(\delta)$ is semidistributive and extremal, it has an action of rowmotion on it, and one that can be computed in slow motion. The join irreducible elements of $\mathrm{Tam}_\nu(\delta)$ can be identified with the boxes of $F_{\delta,\nu}$, and, via this identification, the canonical label of the cover relation $T \lessdot T'$ is the box $b_{T,T'}$ [9]. Hence, the toggles $\tau_b \colon \mathrm{Tam}_\nu(\delta) \to \mathrm{Tam}_\nu(\delta)$ for boxes b in $F_{\delta,\nu}$ perform a (left- or right-) rotation at b when possible. And we have the following description of rowmotion of (δ,ν)-trees in the alt ν-Tamari lattice:

Lemma 1. *Label the boxes of $F_{\delta,\nu}$ as $1,2,\ldots,\ell$ in "Arabic/Hebrew reading order" from right to left and top to bottom. Then rowmotion on $\mathrm{Tam}_\nu(\delta)$ is $\mathrm{Row} = \tau_\ell \circ \cdots \circ \tau_1$. See Figure 6.*

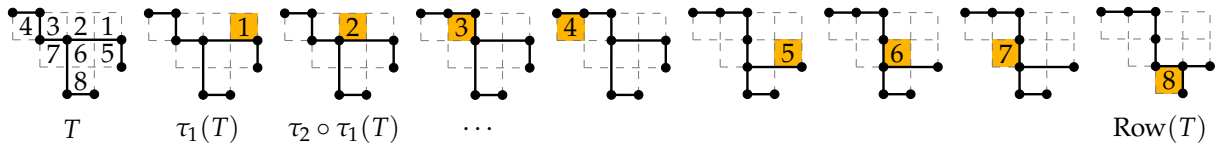

Figure 6: Rowmotion in slow motion for a tree T: $\mathrm{Row}(T) = \tau_8 \circ \cdots \circ \tau_1(T)$. Note that T has ascent boxes 3 and 8, which are exactly the descent boxes of the output $\mathrm{Row}(T)$.

4 The rowmotion-intertwining bijection

We now construct the rowmotion-intertwining bijection between alt ν-Tamari lattices from Theorem 1. So fix a path ν. Note that it suffices to construct the bijection in the case when the two lattices are $\mathrm{Tam}_\nu(\delta)$ and $\mathrm{Tam}_\nu(\widetilde{\delta})$, where $\widetilde{\delta}$ is obtained from δ by decrementing its last nonzero entry by one. Indeed, denoting the bijection in this case by $\varphi \colon \mathrm{Tam}_\nu(\delta) \to \mathrm{Tam}_\nu(\widetilde{\delta})$, we can then define the bijection $\Phi \colon L_1 \to L_2$ between arbitrary alt ν-Tamari lattices L_1 and L_2 as an appropriate combination of these φ's and their inverses. So now we focus on constructing $\varphi \colon \mathrm{Tam}_\nu(\delta) \to \mathrm{Tam}_\nu(\widetilde{\delta})$ in this case.

First let us observe that the right-flushing bijection from ν-Dyck paths to (δ,ν)-trees, depicted in Figure 3, is determined by the property that it preserves the number of nodes at every height. Hence, there is similarly a bijection $\mathrm{flush} \colon \mathrm{Tam}_\nu(\delta) \to \mathrm{Tam}_\nu(\widetilde{\delta})$ which likewise is determined by the property of preserving the number of nodes at every height. This flush map *does not* intertwine rowmotion, but it will be a component of the rowmotion-intertwining bijection $\varphi \colon \mathrm{Tam}_\nu(\delta) \to \mathrm{Tam}_\nu(\widetilde{\delta})$.

Because we work with and compare both $\mathrm{Tam}_\nu(\delta)$ and $\mathrm{Tam}_\nu(\widetilde{\delta})$, we will use tildes to denote maps on the latter. For example, we use $\widetilde{\mathrm{Row}} \colon \mathrm{Tam}_\nu(\widetilde{\delta}) \to \mathrm{Tam}_\nu(\widetilde{\delta})$ to denote rowmotion on this lattice. Also, we let $F := F_{\delta,\nu}$ and $\widetilde{F} := F_{\widetilde{\delta},\nu}$. Suppose that δ_k is the last nonzero entry in δ. Then we let F^{up} denote the part of F above height k and F^{down}

denote the part below height k, and define $\widetilde{F}^{\mathrm{up}}$ and $\widetilde{F}^{\mathrm{down}}$ similarly with respect to $\widetilde{F}$. See Figure 7. Notice that $\widetilde{F}^{\mathrm{up}} = F^{\mathrm{up}}$ and $\widetilde{F}^{\mathrm{down}}$ is obtained from F^{down} by shifting all the boxes one unit right.



Figure 7: The stack polyominoes $F = F_{\delta,\nu}$ and $\widetilde{F} = F_{\widetilde{\delta},\nu}$, together with their upper and lower parts. Here $\nu = (1,2,1,0)$, $\delta = (1,0,0)$, and $\widetilde{\delta} = (0,0,0)$.

We use τ^{up} to denote the composition of all the δ-toggles corresponding to boxes in F^{up} in the same order as rowmotion, and τ^{down} to denote the composition of all the δ-toggles for boxes in F^{down} in that order. For the example in Figures 6 and 7, we have $\tau^{\mathrm{up}} = \tau_7 \circ \cdots \tau_1$ and $\tau^{\mathrm{down}} = \tau_8$. Note that we always have $\mathrm{Row} = \tau^{\mathrm{down}} \circ \tau^{\mathrm{up}}$. We define $\widetilde{\tau}^{\mathrm{up}}$ and $\widetilde{\tau}^{\mathrm{down}}$ similarly for $\widetilde{F}$ and the $\widetilde{\delta}$-toggles.

We now define the rowmotion-intertwining bijection $\varphi\colon \mathrm{Tam}_\nu(\delta) \to \mathrm{Tam}_\nu(\widetilde{\delta})$ as

$$\varphi := \mathrm{flush} \circ \tau^{\mathrm{up}}.$$

See Figure 8 for an example.

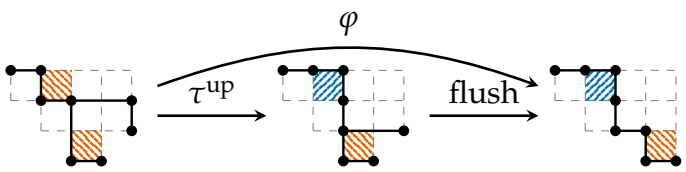

Figure 8: An example of the map φ.

The main thing we have to prove about φ is of course that it intertwines rowmotion:

Theorem 3. *We have* $\widetilde{\mathrm{Row}} \circ \varphi = \varphi \circ \mathrm{Row}$.

The main strategy for proving Theorem 3 is to show that in fact φ intertwines the τ^{up} and τ^{down} individually, that is, that: $\widetilde{\tau}^{\mathrm{up}} \circ \varphi = \varphi \circ \tau^{\mathrm{up}}$ and $\widetilde{\tau}^{\mathrm{down}} \circ \varphi = \varphi \circ \tau^{\mathrm{down}}$. This is illustrated in Figure 9. For full details of the proof of Theorem 3, see [1].

The other statement in Theorem 1, besides the fact that the bijection between alt ν-Tamari lattices commutes with rowmotion, is that it preserves down-degree sums along

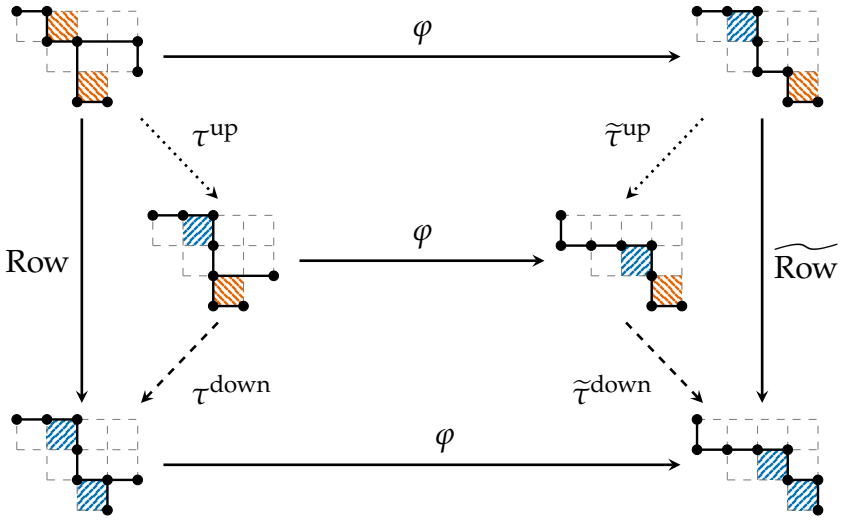

Figure 9: An illustration of Theorem 3: $\widetilde{\text{Row}} \circ \varphi = \varphi \circ \text{Row}$.

rowmotion orbits. To see why φ has this property, first notice that for alt ν-Tamari lattices, down-degree corresponds to number of descent boxes in the tree, and similarly *up-degree* (number of upper covers) corresponds to number of ascent boxes. Also, note that rowmotion converts up-degree to down-degree. Finally, observe that φ has a predictable behavior on the ascent and descent boxes in the upper and lower parts, respectively, and hence preserves number of descent boxes along a rowmotion orbit. Again, see [1] for the details. This completes the proof of Theorem 1.

5 Rational Tamari lattices

For a generic choice of lattice path ν, the alt ν-Tamari lattices are not particularly well behaved. For example, there is no simple product formula for the number of ν-Dyck paths for generic ν. Additionally, rowmotion tends to behave in a rather disorderly fashion on generic alt ν-Tamari lattices. For certain special families of lattice paths ν, however, rowmotion exhibits a remarkably orderly behavior. We study these special families of alt ν-Tamari lattices in this section.

Fix coprime positive integers $a < b$, and let ν be the lowest northeast path that connects the lower-left and upper-right corners of an $a \times b$ rectangle and stays weakly above the diagonal line connecting these two corners. For example, if $(a, b) = (4, 15)$, then ν is the lattice path $\nu = NE^3NE^4NE^4NE^4$, associated to the partition shape $\lambda = (11, 7, 3, 0)$. We call the corresponding set of ν-Dyck paths the set of *rational Dyck paths*. The corresponding ν-Tamari lattices are called the *rational Tamari lattices*. The rational

Tamari lattices were first defined and studied by Préville-Ratelle and Viennot [10].

It is well-known that the number of (a,b)-rational Dyck paths is given by the *rational Catalan number*

$$\mathrm{Cat}(a,b) := \frac{1}{a+b}\binom{a+b}{a}.$$

Notice that when $(a,b) = (n,n+1)$ for an integer $n \geq 1$, we recover the classical Catalan numbers $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$. More generally, when $(a,b) = (n,nm+1)$ for $n,m \geq 1$, these numbers $\mathrm{Cat}(n,nm+1) = \frac{1}{mn+1}\binom{(m+1)n}{n}$ are called *Fuss–Catalan numbers* and the corresponding Tamari lattices are the *m-Tamari lattices* of Bergeron and Préville-Ratelle [5].

Rowmotion on the rational and m-Tamari lattices was previously studied by Thomas and Williams [12] and Defant and Liu [7], who made a number of conjectures about them, which we are able to prove with the help of Theorem 1. Namely:

Theorem 4. *For any coprime integers $a < b$, the order of rowmotion acting on any alt (a,b)-rational Tamari lattice L divides $a+b-1$. Moreover, the triple $(L, \mathbb{Z}/(a+b-1)\mathbb{Z}, \mathrm{Cat}_q(a,b))$ exhibits cyclic sieving, where $\mathrm{Cat}_q(a,b) := \frac{1}{[a+b]_q}\binom{a+b}{a}_q$ is the rational q-Catalan number and the generator of the cyclic group acts on L as* Row.

Theorem 5. *For any $n,m \geq 1$, and for any alt m-Tamari lattice L, the down-degree statistic* ddeg *is homomesic under the action of* Row *on L, with average $\frac{m(n-1)}{m+1}$.*

The key to proving both Theorem 4 and Theorem 5 is to use Theorem 1 to transfer to either the distributive lattice case or the m-Tamari case and then use prior results [12, §7.6] and [7, Theorem 5.14] on these lattices.

6 Future directions

Very recently, von Bell and Ceballos introduced the family of *cross Tamari lattices* [4]. While the alt ν-Tamari lattices depend on a stack polyomino shape, the cross Tamari lattices depend on a *moon polyomino* shape. Here a moon polyomino is a shape that can be obtained from a Young diagram by permuting both the columns and the rows, so that when read from left to right the column lengths increase and then decrease, and that when read from top to bottom the row lengths similarly increase and then decrease. In this way, moon polyominoes are more general than stack polyominoes, and similarly the cross Tamari lattices are more general than the alt ν-Tamari lattices. The cross Tamari lattices are all still semidistributive and extremal, hence carry an action of rowmotion.

We conjecture that the invariance of rowmotion extends to cross Tamari lattices:

Conjecture 1. *Let L_1 and L_2 be two cross Tamari lattices associated to moon polyominoes that are related by a sequence of row and column permutations. Then rowmotion behaves the same (in the sense of Theorem 1) for L_1 and L_2.*

For an example of Conjecture 1, see Figure 10. However, we note that the bijection φ from Section 4, or at least a naive extension of it, does not work for cross Tamari lattices.

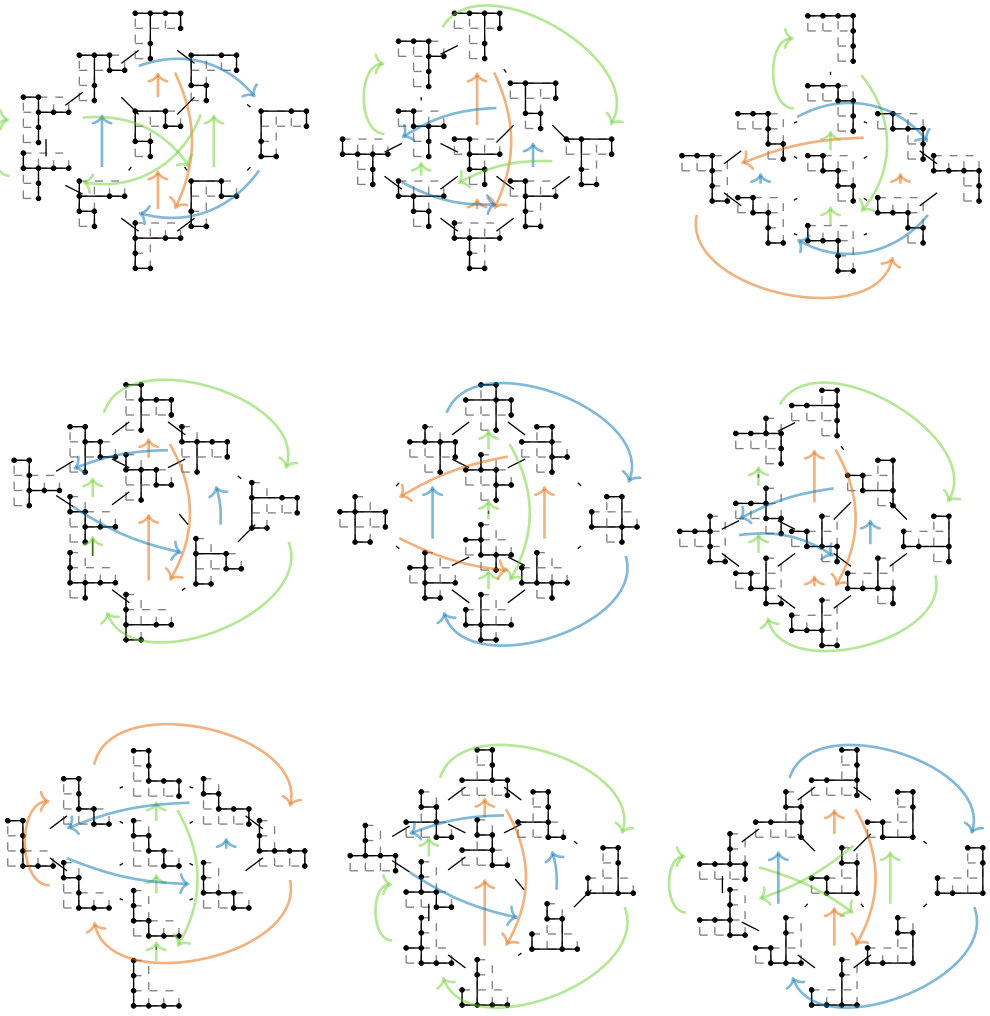

Figure 10: Rowmotion orbits for all cross Tamari lattices corresponding to moon polyominoes that are permutations of the Young diagram $(3,1,1)$.

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AI disclosure

We did not use any AI tools at any point during our research and writing processes.