MODULAR UPHO LATTICES

SAM HOPKINS

Abstract. I present a bonus problem for the 2025 BIRS lattice theory workshop.

A poset $\mathcal{P}$ is called *upho* (short for "upper homogeneous") if every principal order filter (i.e., induced subposet of the form $\{q \in \mathcal{P} \colon q \geq p\}$ for some $p \in \mathcal{P}$) is isomorphic to the original poset $\mathcal{P}$. We assume that all upho posets are finite type $\mathbb{N}$-graded, meaning that we can write our poset $\mathcal{P}$ as $\mathcal{P} = \sqcup_{i=0}^{\infty} P_i$ such that every maximal chain is of the form $x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots$ with $x_i \in P_i$, and such that each P_i is finite. Then we can form the rank-generating function $F(\mathcal{P};x) := \sum_{i=0}^{\infty} \#P_i \, x^i$.

Now let $\mathcal{L}$ be an upho lattice. Its *core* L is defined to be the interval from the minimum of $\mathcal{L}$ to the join of its atoms. By definition, this core is a finite graded lattice. In [3] it is shown that $F(\mathcal{L};x) = \chi^*(L;x)^{-1}$, where $\chi^*(L;x)$ is the (reciprocal) characteristic polynomial of the finite lattice L. So the core determines a lot about the upho lattice. In [4, 5] we pursued the problem of classifying upho lattices, using their cores. But this classifcation is very far from achieved.

If $\mathcal{L}$ is a *modular* upho lattice, then its core L is a finite modular lattice for which the maximum is the join of atoms. Note that, for a finite modular lattice, having the maximum be the join of atoms is equivalent to being complemented. And it is a classical result (due to Veblen, Birkhoff, Tits, etc.) that there is a correspondence between the following structures:

- finite complemented modular lattices;
- finite projective geometries;
- finite spherical buildings.

For example, for a prime power q and integer $n \geq 1$, the lattice $B_n(q)$ of $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$ is a finite complemented modular lattice. And if the lattice is indecomposable as a product and has rank at least 4, these are the only ones.

Example. Let $n \geq 1$ and let p be a prime. The lattice of subgroups of $\mathbb{Z}^n$ of index a power of p (ordered by reverse inclusion) is a modular upho lattice with core $B_n(p)$. The case $n = 2$ and $p = 2$ is depicted in Figure 1.

Example. Let $n \geq 1$ and let q be a prime power and let R be a discrete valuation ring with residue field $\mathbb{F}_q$. The lattice of full rank submodules of R^n is a modular upho lattice with core $B_n(q)$. The previous example is the case $R = \mathbb{Z}_{(p)}$.

These examples are due to Stanley [7] and he conjectured (see [1, Conjecture 1.1]) that they are essentially the only ones. As it turns out, Stanley's conjecture is not quite right, basically because of non-Desarguesian projective planes.

Date: December 31, 2024.

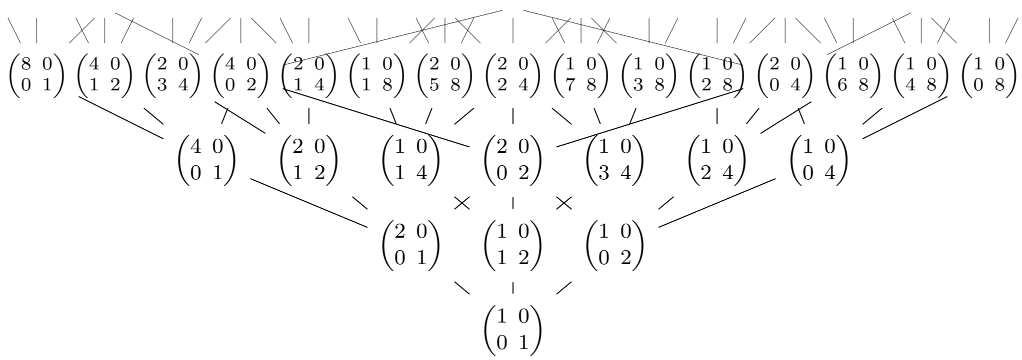

FIGURE 1. Subgroups of $\mathbb{Z}^2$ of index a power of 2, ordered by reverse inclusion.

More precisely, given a (naturally colored) affine building, Hirai explained in [2] how to produce a corresponding modular lattice. If the group of color-rotating automorphisms acts transitively on the vertices of the affine building, then any principal order filter in the corresponding lattice constructed by Hirai will be a modular upho lattice.

Moreover, Radu constructed in [6] an $\widetilde{A}_2$-building, who group of color-rotating automorphisms acts transitively on its vertices, and whose residue planes are all isomorphic to the Hughes projective plane of order 9, which is non-Desarguesian. Hence, the corresponding modular upho lattice cannot come from a discrete valuation ring in the way Stanley predicted.

Nevertheless, we suspect that Stanley's intuition was essentially correct. Namely:

Problem. *Show that all modular uhpo lattices come from (sufficiently symmetric) affine buildings in the way we have outlined above.*

Remark. The situation for *distributive* upho lattices is much more trivial. It follows easily from the representation theorem for locally finite distributive lattices that the only distributive upho lattices are $\mathbb{N}^n$.

References

[1] Y. Gao, J. Guo, K. Seetharaman, and I. Seidel. The rank-generating functions of upho posets. *Discrete Math.*, 345(1):Paper No. 112629, 14, 2022.
[2] H. Hirai. Uniform modular lattices and affine buildings. *Adv. Geom.*, 20(3):375–390, 2020.
[3] S. Hopkins. A note on Möbius functions of upho posets. *Electron. J. Combin.*, 29(2):Paper No. 2.39, 6, 2022.
[4] S. Hopkins. Upho lattices I: examples and non-examples of cores. `arXiv:2407.08013`, 2024.
[5] S. Hopkins and J. Lewis. Upho lattices II: ways of realizing a core, 2025. In preparation.
[6] N. Radu. A lattice in a residually non-Desarguesian $\widetilde{A}_2$-building. *Bull. Lond. Math. Soc.*, 49(2):274–290, 2017.
[7] R. P. Stanley. From Stern's triangle to upper homogeneous posets. Talk transparencies, `https://math.mit.edu/~rstan/transparencies/stern-ml.pdf`, 2020.

Email address: `samuelfhopkins@gmail.com`

DEPARTMENT OF MATHEMATICS, HOWARD UNIVERSITY, WASHINGTON, D.C. USA